PHYS 206 Lecture 9

9.1) Angular Momentum

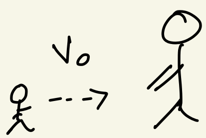
9.2) Cross product

9.3) Torque and moment of inertia

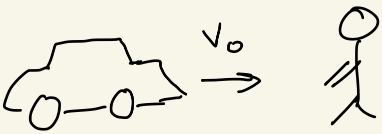
(*) Previously, we defined the "linear momentum" as

$$\boxed{\vec{p} = m\vec{v}}$$

This quantity represents how difficult it is to change an object's path:

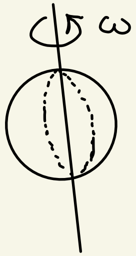


Child running at you → easy to push over (I assume).



Car moving at you → get out of its way.

(*) Angular momentum is a similar concept, except for rotating objects:



How difficult is it to stop an object that is rotating?

* Two common types of angular momentum:

(1) <u>Orbital</u>: object rotating about a point.

→ Earth rotating around Sun

→ Swing object in circular path by a string

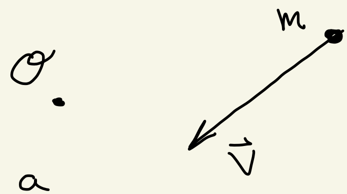
(2) <u>Spin</u>: solid object rotating about axis.

→ Spinning basketball

→ Rotating disk

* These two types of angular momentum are fairly intuitive, but there are several <u>unintuitive</u> <u>aspects</u> → so be careful!

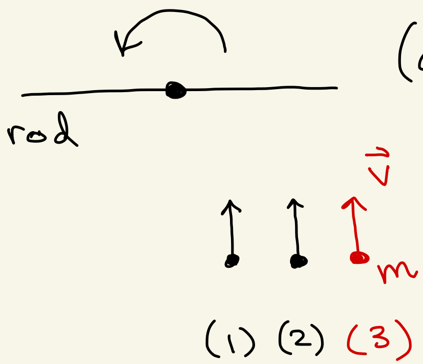
Unintuitive example:



An object moving in a straight line some distance from the origin has angular momentum!

Example: Consider a rod that is free to rotate without friction about an axis perpendicular to its center. If you throw a rock at different points away from the axis, which will cause rod to rotate the fastest?

(overhead view)



⊛ Rock thrown from position (3) will cause rod to rotate fastest.

⊛ Increasing m or v will also result in rod rotating faster.

Definition of angular momentum:

$$\boxed{\vec{L} = \vec{r} \times \vec{p}}$$

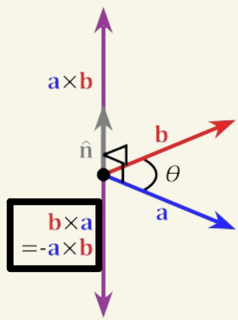
Vector "cross product" largest when $\vec{r}$ and $\vec{p}$ are at 90°.

Larger distance from axis gives larger L.

Larger linear momentum gives rise to larger L.

9.2) Cross product

⊛ Type of vector product that produces an output vector.



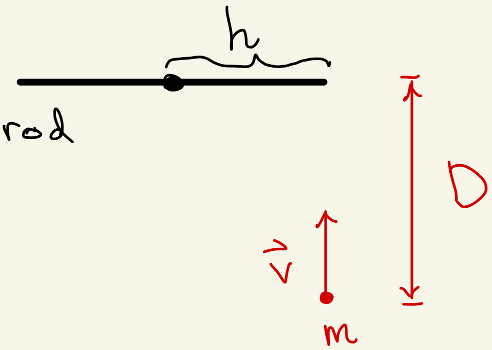
⊛ The cross product of two vectors $\vec{a}$ and $\vec{b}$ has

(i) Magnitude $\boxed{|\vec{a} \times \vec{b}| = ab \sin\theta}$

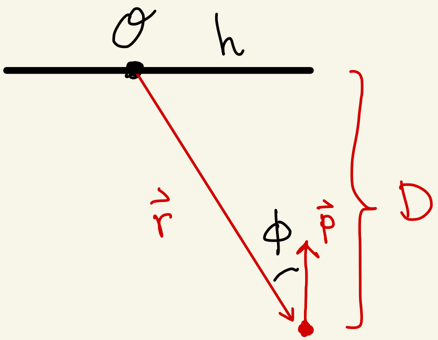
(ii) Direction given by right-hand-rule:

Point fingers in direction of 1st vector, curl them toward 2nd vector, then thumb gives direction of cross product vector that must be perpendicular to both $\vec{a}$ and $\vec{b}$.

* Let's calculate $\vec{L} = \vec{r} \times \vec{p}$ for the rock example above at an arbitrary distance D from hitting the rod:



Always choose origin at rotation axis



$$\sin\phi = h/r$$

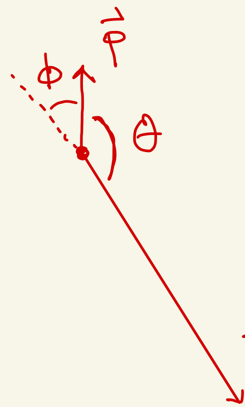
$$\vec{p} = m\vec{v}$$

$$r = \sqrt{h^2 + D^2}$$

$$\vec{L} = \vec{r} \times \vec{p} \longrightarrow L = rp\sin\theta$$

$$\Rightarrow L = rp\sin(\pi - \phi)$$

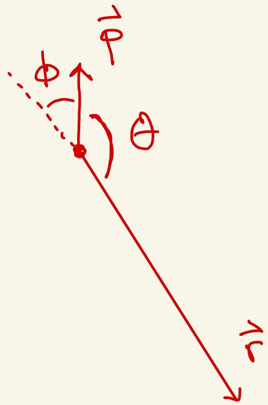
$$L = rp\sin\phi = rp\,(h/r)$$



$$L = ph$$

Constant along entire path!!

Direction of $\vec{L}$ vector:



Pointing fingers toward $\vec{r}$ and curling them toward $\vec{p}$ would force your thumb to point <u>out</u> <u>of</u> <u>page</u>.

(*) Note that cross product <u>direction</u> also does not change as mass m moves upward.

Angular momentum of a freely moving object is constant (conserved)

9.3) Torque and moment of inertia

- Recall that forces cause acceleration

$$\vec{F} = m\vec{a}$$

or equivalently, forces causes changes in momentum $\vec{F} = m\frac{d\vec{v}}{dt} = \frac{d\vec{p}}{dt}$.

- Analogously, "torques" cause angular acceleration, or equivalently torques cause a change in angular momentum:

$$\boxed{\vec{\tau} = \frac{d\vec{L}}{dt}}.$$

- With this definition, we see that

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r}\times\vec{p}) = \frac{d\vec{r}}{dt}\times\vec{p} + \vec{r}\times\frac{d\vec{p}}{dt}$$

$$= \underbrace{\vec{v}\times\vec{p}}_{0} + \vec{r}\times\vec{F}$$

$$= \vec{r}\times\vec{F}.$$

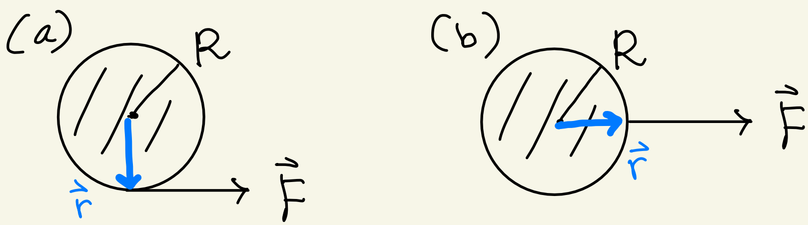
Call this quantity "torque".

Definition

Torque: $\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$

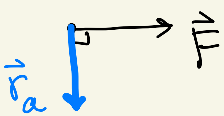
$\vec{r}$ is from the origin to position where $\vec{F}$ is applied.

Example: Consider the same force $\vec{F}$ applied at two different points on a wheel



Due to the position $\vec{r}$ at which $\vec{F}$ is applied, in case (a) the wheel will start to rotate but in case (b) it will not

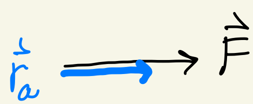
$\vec{\tau}_a = \vec{r}_a \times \vec{F}$



$\vec{\tau}_a = RF\sin 90^\circ$

$= RF$ out of page

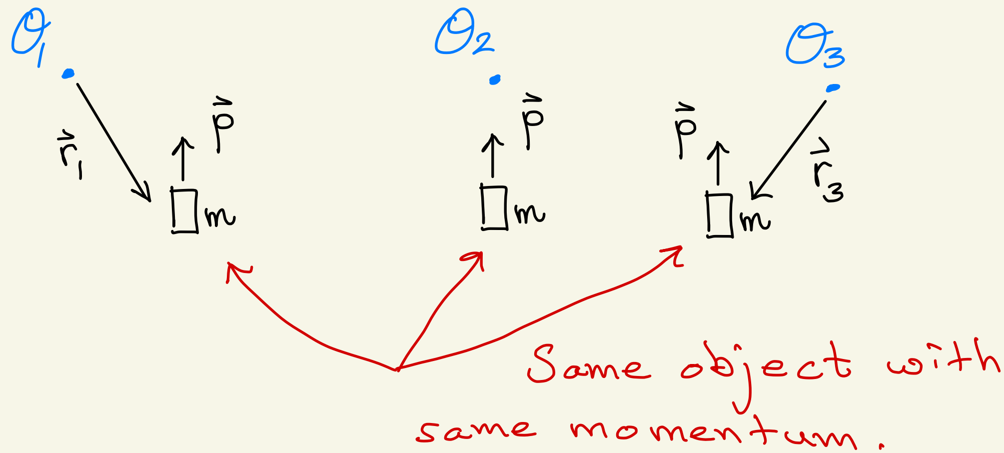
$\vec{\tau}_b = \vec{r}_b \times \vec{F}$



$\vec{\tau}_b = RF\sin 0$

$= 0$

(*) Crucial point about both angular momentum and torque is that they depend sensitively on the location of the origin. (Many other vectors $\vec{v}$, $\vec{a}$, $\vec{p}$, $\vec{F}$, etc. do not depend on origin.)



O_1 : $\vec{L}$ is nonzero and out of page $\odot$

O_2 : $\vec{L}$ is zero

O_3 : $\vec{L}$ is nonzero and into page $\otimes$

(*) Angular analogue to $\vec{F} = m\vec{a}$:

$$\vec{r} \times \vec{F} = \vec{r} \times (m\vec{a}) = m(r\hat{\imath}_r) \times (a_r\hat{\imath}_r + a_\theta\hat{\imath}_\theta)$$

$$\Rightarrow \vec{\tau} = mr\left(\alpha r + 2\omega\frac{dr}{dt}\right)[\hat{\imath}_r \times \hat{\imath}_\theta] \qquad \boxed{\hat{\imath}_r \times \hat{\imath}_r = 0}$$

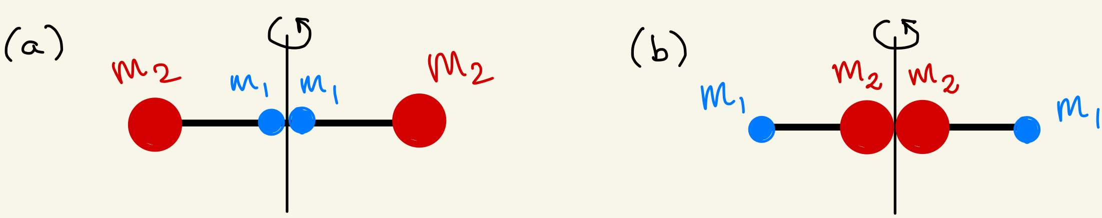
For purely angular motion ($dr/dt = 0$) :

$$\boxed{\tau = mr^2\alpha \equiv I\alpha}$$

"Moment of inertia"

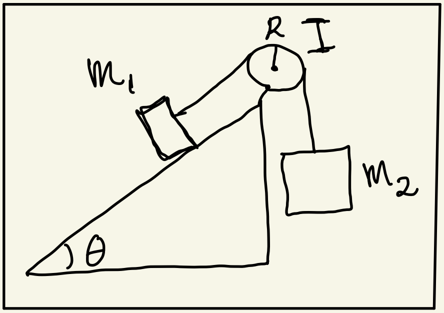
(*) The moment of inertia represents how difficult it is to get an object to rotate. Large $I \Rightarrow$ difficult to rotate.

<u>Example</u>: Which of the two configurations of total mass $2(m_1+m_2)$ would be more difficult to rotate?



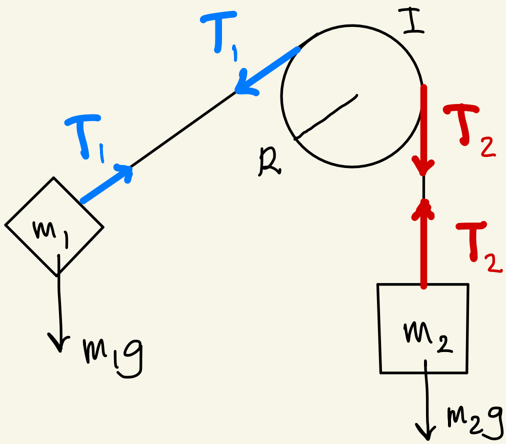
Hopefully your intuition says "(a)".

Example: Consider two blocks connected by a massless string and large pulley with radius R and moment of inertia I. If the ramp is frictionless and $m_2 >> m_1$, find the acceleration of m_2 and the angular acceleration of the pulley.



Solution: The pulley will begin to rotate only if there is a net external torque.

⊛ Key Point → For a massive pulley, the tension in the string must be different on the two sides.



$$T_1 \neq T_2$$

⊛ Since massive pulley rotates, we need an extra equation of motion:

(1) $m_2 g - T_2 = m_2 a$

(2) $T_1 - m_1 g\sin\theta = m_1 a$

$\alpha = \frac{a}{R}$

(3) $\sum \tau = I\alpha \Rightarrow RT_2 - RT_1 = I\alpha$

$$\Rightarrow RT_2 - RT_1 = I\frac{a}{R}$$

There are 3 equations and 3 unknowns (T_1, T_2, a):

$$T_2 = m_2(g-a)$$

$$T_1 = m_1(g\sin\theta + a)$$

$$\Rightarrow R(T_2 - T_1) = I\frac{a}{R}$$

$$\Rightarrow m_2 g - m_2 a - m_1 g\sin\theta - m_1 a = \frac{I}{R^2}a$$

$$\Rightarrow \frac{m_2 g - m_1 g\sin\theta}{m_1 + m_2 + \frac{I}{R^2}} = a$$

* The moment of inertia is also related to the angular momentum of a rigid body:

$$\boxed{L = I\omega}$$

Compare to $\vec{p} = m\vec{v}$.

<u>Example</u>: Consider a solid rod of length L that is free to rotate about its center with moment of inertia I_o. If a bug with mass m crawls from the rotation axis to the edge of the rod according to $r(t) = \alpha t$, find the total moment of inertia as a function of time.

<u>Solution</u>: $I_{total} = I_o + mr^2$

$$I_{total} = I_o + m(\alpha t)^2$$

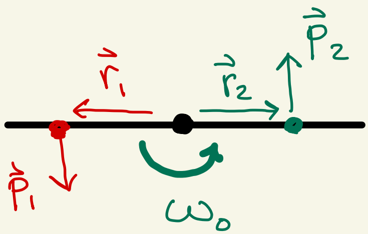
$$= I_o + m\alpha^2 t^2$$

If the rod rotates at a constant angular velocity ω_0, how does its angular momentum vary in time?

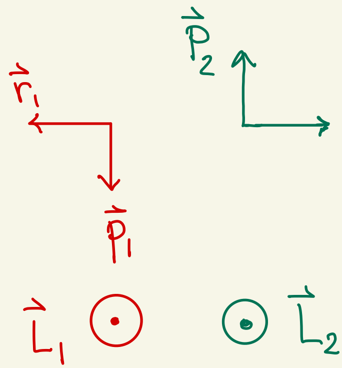
Answer: $L = I\omega$

$$= (I_0 + m\alpha^2 t^2)\,\omega_0$$

What about direction of $\vec{L}$?



(overhead view)



$\vec{L}$ is <u>out</u> <u>of</u> <u>page</u> when you look at any individual contribution to the total $\vec{L}$.

Rule for rigid body: Curl right hand fingers in direction of rotation, then thumb points in direction of $\vec{L}$.