8.) Time-independent circuits

8.1) Kirchhoff's Laws
8.2) Resistors in parallel and series
8.3) Power in circuits

8.1) Kirchhoff's Laws

⊛ There are two fundamental rules for analyzing circuits known as "Kirchhoff's Laws"

⊛ Important Point: These are not new laws of physics! Instead, they are just basic rules of electric charges and forces applied to circuits.

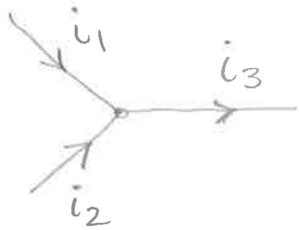
Rule 1: The current that enters a junction must be equal to the current that leaves a junction.

Yes, this is as obvious as it sounds.

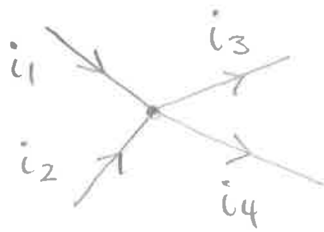
Charge must flow somewhere, and in a steady-state circuit charge cannot keep accumulating somewhere

Mathematically $\oint \vec{J} \cdot d\vec{A} = 0$ (current conservation)

Examples:



$$i_1 + i_2 = i_3$$



$$i_1 + i_2 = i_3 + i_4$$

Here the arrows denote the direction of current flow.

> Rule 2: If you walk along a circuit path and add up all the changes in electric potential, you must get 0 if you start and end at the same point.

(*) This just follows from the fact that the electric force is conservative:

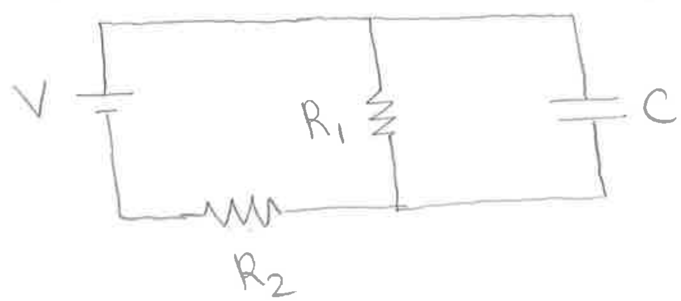
$$\oint \vec{F} \cdot d\vec{r} = 0$$

$$V(\vec{r}_1) - V(\vec{r}_1) = -\int_{\vec{r}_1}^{\vec{r}_1} \vec{E} \cdot d\vec{r} = 0$$

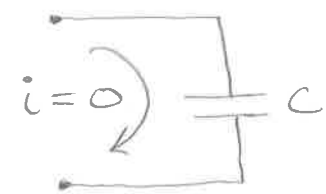
This is true independent of path!

(*) Closed paths form loops → "Kirchhoff's Loop Rule"

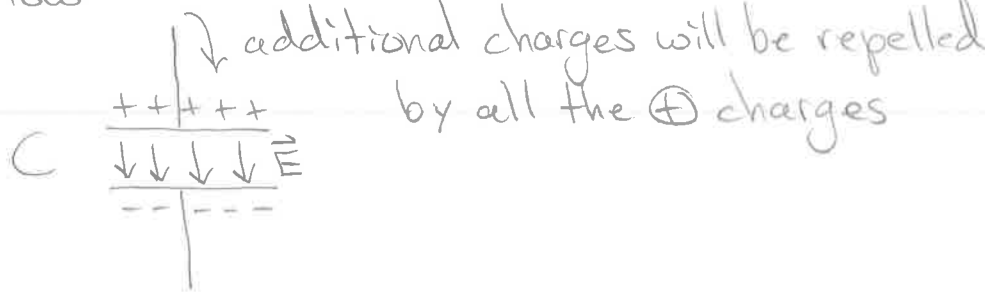
Examples: Consider the circuit below that has been connected for a very long time.



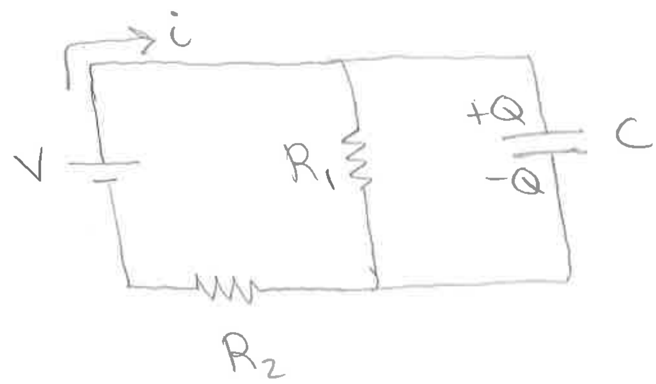
(*) Crucial Point #1 : Charge does not flow at any point along the rightmost wire



This is because after a very long time enough charge has accumulated on the capacitor so that it blocks any extra flow



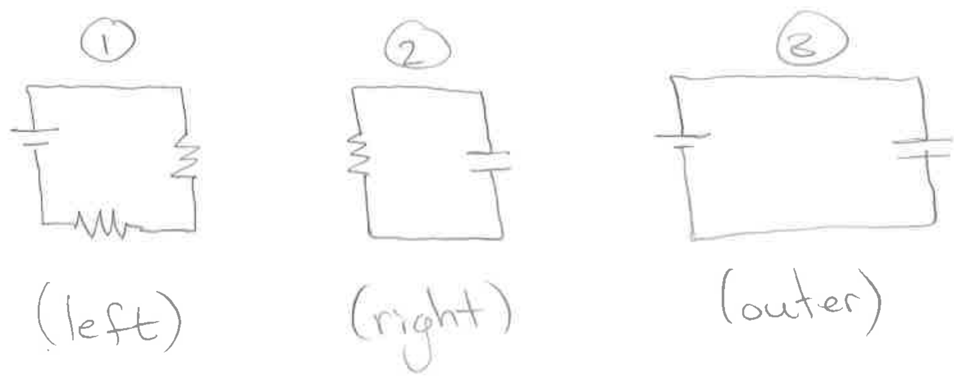
Therefore, current only flows through the left rectangular section of wire.



(*) Note that there are 2 unknowns: i and Q on capacitor. Let's find them using Kirchhoff's 2nd Rule.

(*) Crucial Point #2: In applying Kirchhoff's Loop Rule, pretend you are an observer walking along the wire and just recording what happens. In particular, the "loops" in Kirchhoff's Loop Rule do not refer to current loops. Loops can include sections of wire where there is no current.

In the above figure there are three main loops:



Left loop: Start underneath battery and go clockwise

$$+V - iR_1 - iR_2 = 0$$

$+V$: Voltage gain going from ⊖ to ⊕ terminal.

$-iR_1 - iR_2$: Voltage drops going across a resistor in the direction of current.

$$\Rightarrow \boxed{i = \frac{V}{R_1 + R_2}}$$

Right loop: Start underneath resistor and go clockwise

$$+iR_1 - \frac{Q}{C} = 0$$

$+iR_1$: Voltage gain across a resistor in the direction opposite the current

$-\frac{Q}{C}$: Voltage drop going from the $+Q$ to the $-Q$ sides of a capacitor

$$\Rightarrow \boxed{Q = iCR_1 = \left(\frac{V}{R_1+R_2}\right)CR_1 = VC\left(\frac{R_1}{R_1+R_2}\right)}$$

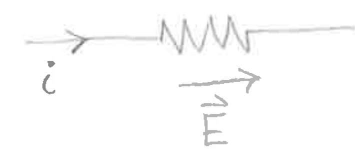
Outer loop: Start above battery and go counterclockwise

$$-V + iR_2 + \frac{Q}{C} = 0$$

$$\Rightarrow Q = (V - iR_2)C = \left(V - \frac{V}{R_1+R_2}R_2\right)C$$

$$= VC\left(1 - \frac{R_2}{R_1+R_2}\right) = VC\left(\frac{R_1}{R_1+R_2}\right)$$ same ✓

To Summarize

- (*) Voltage drops across resistors in direction of current flow (

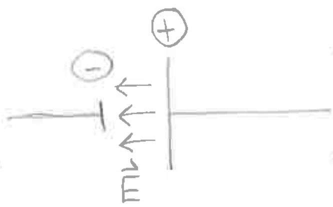
) because $\vec{E}$ points in direction of current

- (*) Voltage drops from ⊕ to ⊖ sides of capacitor

(

) because $\vec{E}$ points in that direction

- (*) Voltage gain from ⊖ to ⊕ terminal of battery

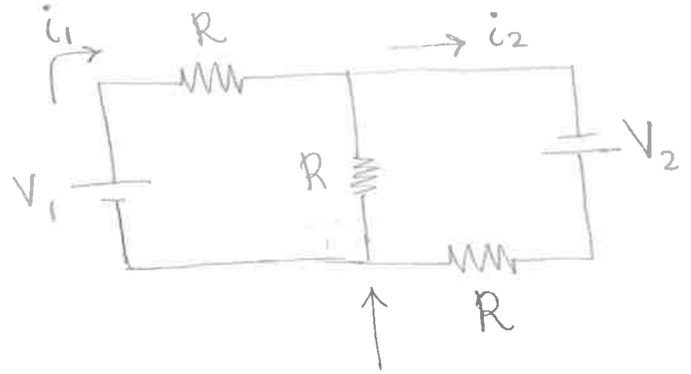
(

) because $\vec{E}$ points in opposite direction

- (*) Obtaining the correct signs (±) is crucial !

Important Point : Sometimes it is not clear in which direction a current flows or which side of a capacitor is positively charged. Do not stress out, just make your best guess and stick with it. If you guessed wrong, the current or charge you calculate will just be a negative number.

Example:



Does current flow up or down through middle segment?

Just make a choice. You must make a choice!

I will choose downward.

then $i_1 = i_2 + i_3$.

Loop Rules:

① Left loop, clockwise path:

$$V_1 - i_1 R - i_3 R = 0$$

$$V_1 = (i_1 + i_3) R$$

② Right loop, clockwise path:

$$V_2 - i_2 R + i_3 R = 0$$

$$V_2 = (i_2 - i_3) R$$

Since $i_1 = i_2 + i_3$ we find

$$V_1 = (i_1 + i_3)R$$
$$V_1 = (i_2 + 2i_3)R$$

Combine with

$$V_2 = (i_2 - i_3)R$$

by subtracting $V_1 - V_2 = 3i_3R$

$$\Rightarrow \boxed{i_3 = \frac{V_1 - V_2}{3R}}$$

Note that if $V_1 > V_2$ then i_3 is positive (and it indeed flows downward).

But if $V_2 > V_1$ then i_3 is negative (which means that in reality i_3 flows opposite to our choice, or upward)

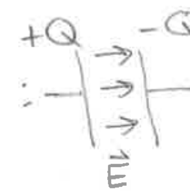
(*) Same is true of capacitors. You must make a choice about which side holds + charge and which side holds − charge.

5 steps to solve circuit diagrams for time-independent systems

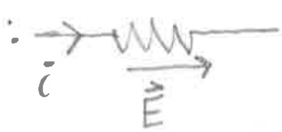
① Identify capacitors (Why? Because current is 0 through a capacitor and any other elements in series with it after a long time).

② Assume charges $\pm Q_1, \pm Q_2, \ldots$ on capacitors in the circuit (you must specify $\pm$ orientation on the plates) as well as currents $i_1, i_2, \ldots$ through sections of the circuit that are in series (you must specify current directions).

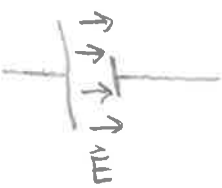
③ From $\pm Q$ orientation determine direction of $\vec{E}$:



From direction of i determine $\vec{E}$ through resistors :



From battery orientation determine $\vec{E}$:



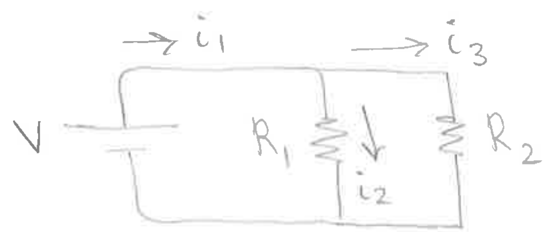
④ Enforce current conservation at each junction : $i_{in} = i_{out}$.

Note that you must use these equations if they exist.

⑤ Form closed paths and set $\Delta V_{total} = 0$. If path direction is parallel to $\vec{E}$ in step ③ $\Rightarrow$ potential drop $(-)$. Otherwise, potential gain $(+)$. $|\Delta V_c| = \frac{Q}{C}$ and $|\Delta V_R| = iR$.

8.2) Resistors in parallel and series

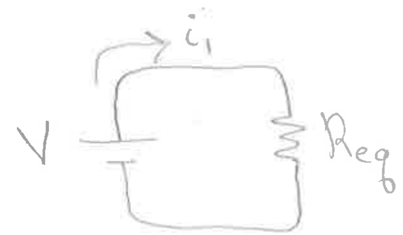
(*) Since resistance decreases with increasing cross sectional area ($R = \frac{\rho L}{A}$), adding resistors in parallel should decrease the effective resistance :



Left Loop: $V - i_2 R_1 = 0 \Rightarrow i_2 = V/R_1$

Outer Loop: $V - i_3 R_2 = 0 \Rightarrow i_3 = V/R_2$

Current conservation: $i_1 = i_2 + i_3$



$$V = i_1 R_{eq}$$

$$V = (i_2 + i_3) R_{eq}$$

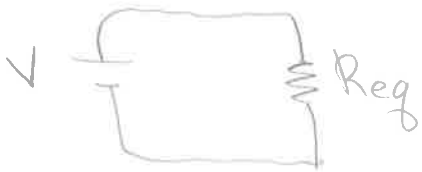
$$V = \left(\frac{V}{R_1} + \frac{V}{R_2} \right) R_{eq}$$

Resistors in parallel

$$\boxed{\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}}$$

(*) Similarly, since resistance increases with increasing length of the resistor ($R = \frac{\rho L}{A}$), adding resistors in series should increase the effective resistance :

$V = i(R_1 + R_2)$



$V = iR_{eq}$

$\Rightarrow \quad i(R_1+R_2) = i\,R_{eq}$

Resistors in Series $\quad \boxed{R_{eq} = R_1 + R_2}$

8.3) Power in circuits

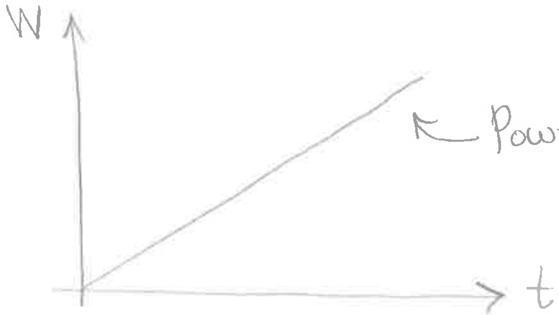
Recall that mobile charges travel with a roughly constant drift velocity $\vec{u}$ in identical sections of wire.

(*) An electron leaving one end of the battery starts with speed u and at the other side ends with same speed u.

Question: Where did all of the energy it gained by lowering its potential energy qV go? It did not go into increasing its kinetic energy.

(*) It must be lost to the rest of the circuit in the form of light and heat.

Once circuit reaches a steady state, power output is constant



← Power $P = \frac{dW}{dt}$ [units of Watts = $\frac{\text{Joules}}{\text{sec}}$]

W is the work done on all charges

(*) The work done on a single charge is just $q\Delta V$.

Power in Circuit : $P = \frac{dW}{dt} = \frac{d}{dt}(q\Delta V)$

$$P = \Delta V \frac{dq}{dt}$$

$$\boxed{P = (\Delta V) I}$$

$V = IR$ → $P = I^2 R$

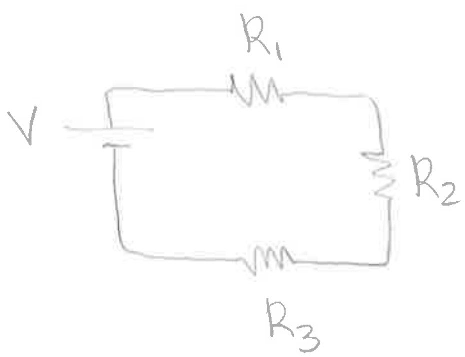
$I = V/R$ → $P = \frac{V^2}{R}$

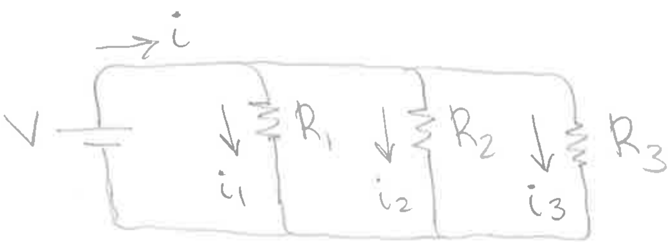
} Alternate forms

Question : Does a battery supply a constant power to a circuit, independent of what is connected to it?

Answer: No! Batteries supply only a constant voltage. The current through the battery can change.

Example: Given resistors $R_1 > R_2 > R_3$, rank the power output through each in the two circuits below.





$$V = i(R_1 + R_2 + R_3)$$

$$i = \frac{V}{R_1 + R_2 + R_3}$$

$$P_1 > P_2 > P_3$$

$$i^2 R_1 > i^2 R_2 > i^2 R_3$$

$$V = i_1 R_1 = i_2 R_2 = i_3 R_3$$

$$P_1 < P_2 < P_3$$

$$\frac{V^2}{R_1} < \frac{V^2}{R_2} < \frac{V^2}{R_3}$$

exact opposite!

Power output can be good (light bulbs, toasters, etc.) or bad (overheating laptop, hot light bulbs, etc.)