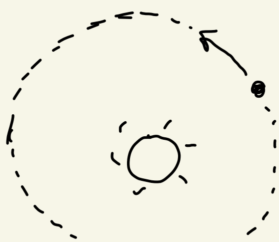
Lecture 8

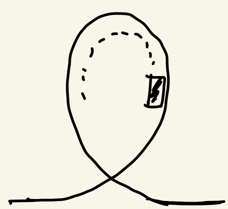
8.1 Intro to rotational motion

* Until now we have not considered objects that are rotating or revolving.

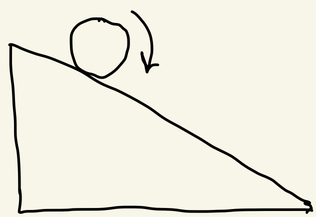
Examples:



Earth rotating around Sun



Roller coaster



Ball rolling down incline

* To analyze these motions, we need to introduce many <u>angular</u> counterparts to quantities we have already discussed.

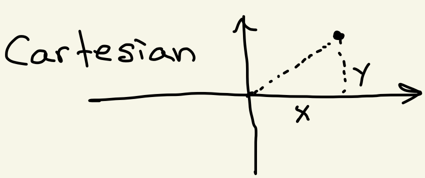
Examples: Angular velocity, angular acceleration, torque ("angular force"), moment of inertia ("rotational mass"), rotational kinetic energy, angular momentum.

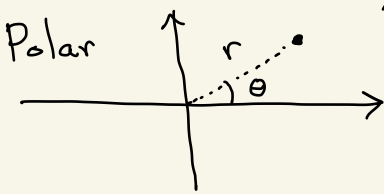
* The mathematical expression of physics laws dealing with angular and rotational kinematics is greatly simplified with "polar coordinates".

8.2 Polar coordinates

* Although many of you have encountered polar coordinates before (e.g., defining a function in polar coordinates) there are some tricky aspects related to vectors in polar coordinates.

Easy: position in polar coordinates





Position defined by r and θ.

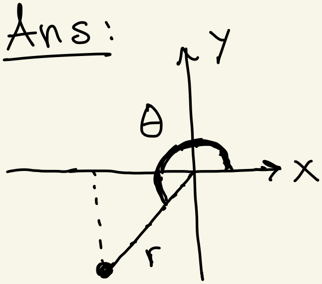
Relation between (x,y) and (r,θ):

$$\boxed{r = \sqrt{x^2+y^2}\ , \ \theta = \tan^{-1}(y/x)}$$

Multivalued, so be careful.

Example: Consider an object located at the position $x = -3\,m$, $y = -3\,m$. What is the position of the object in polar coordinates?

Ans:



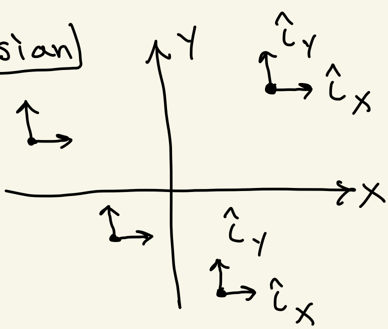
$$r = \sqrt{x^2+y^2} = \sqrt{(-3\,m)^2+(-3\,m)^2}$$
$$= 3\sqrt{2}\,m$$

$$\theta = \pi + \tan^{-1}(1) = \frac{5}{4}\pi$$

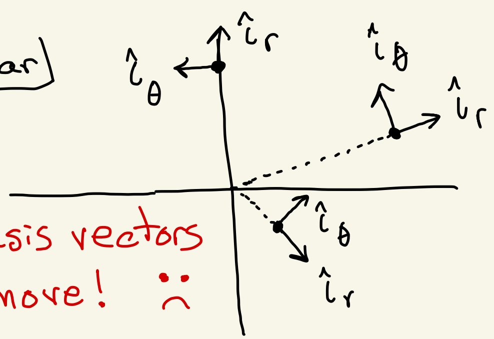
Difficult: vectors in polar coordinates

(*) Basis vectors defined by moving in direction of one coordinate, keeping all others fixed.

Cartesian

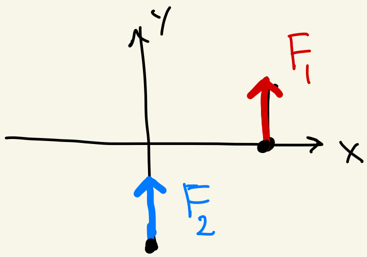


Polar



Basis vectors move! :(

<u>Question</u>: What are the vectors below in spherical coordinates?



<u>Solution</u>: At the location of $\vec{F}_1$, the basis vectors are $\hat{\imath}_\theta$, $\hat{\imath}_r$. Therefore, $\vec{F}_1 = F_1 \hat{\imath}_\theta$.

Likewise at $\vec{F}_2$: $\hat{\imath}_\theta$, $\hat{\imath}_r$. Therefore, $\vec{F}_2 = -F_2 \hat{\imath}_r$.

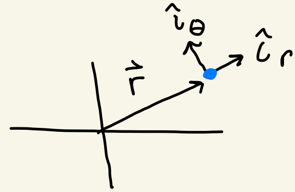
⊛ Interestingly, these are <u>identical</u> <u>vectors</u>!!!

<u>Question</u>: What is the position vector in polar coordinates?

<u>Wrong Ans</u>: $\vec{r} \neq r\hat{\imath}_r + \underbrace{\theta}\hat{\imath}_\theta$

This doesn't have units of length.

Correct Ans : $\boxed{\vec{r} = r\,\hat{\imath}_r}$

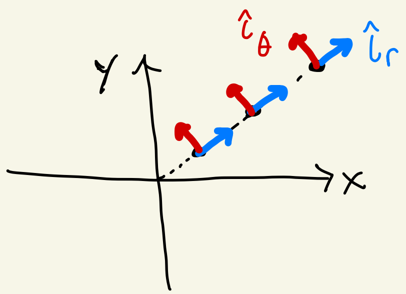


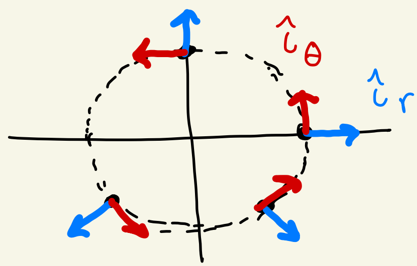
How is this possible? Position vector must depend on both r and θ, right?

→ Yes! That is because $\hat{\imath}_r$ depends on θ:

$$\boxed{\begin{aligned} \hat{\imath}_r &= \cos\theta\,\hat{\imath}_x + \sin\theta\,\hat{\imath}_y \\ \hat{\imath}_\theta &= -\sin\theta\,\hat{\imath}_x + \cos\theta\,\hat{\imath}_y \end{aligned}}$$

The basis vectors do not change with r, only with θ:





$$\Rightarrow \quad \vec{r} = r\,\hat{\imath}_r = \underbrace{r\cos\theta}_{x}\,\hat{\imath}_x + \underbrace{r\sin\theta}_{y}\,\hat{\imath}_y \,.$$

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}} \quad \Longleftrightarrow \quad \boxed{\begin{aligned} r &= \sqrt{x^2+y^2} \\ \theta &= \tan^{-1}(y/x) \end{aligned}}$$

$\boxed{8.3}$ Velocity and acceleration in polar coordinates

(*) Question #1 of Exam 3 is always the same:

Find $\vec{v}$ and $\vec{a}$ in polar coordinates.

Solution: You can only take 3 things as given

1) $\vec{r} = r\,\hat{i}_r$
2) $\hat{i}_r = \cos\theta\,\hat{i}_x + \sin\theta\,\hat{i}_y$
3) $\hat{i}_\theta = -\sin\theta\,\hat{i}_x + \cos\theta\,\hat{i}_y$

The rest you must prove as follows.

Definition: $\vec{v} = \dfrac{d\vec{r}}{dt} = \dfrac{d}{dt}(r\,\hat{i}_r)$

Key point is that both r and $\hat{i}_r$ can depend on time.

Need $\dfrac{d}{dt}(\hat{i}_r)$ and soon also $\dfrac{d}{dt}(\hat{i}_\theta)$:

(a) $\dfrac{d}{dt}(\hat{i}_r) = \dfrac{d}{dt}(\cos\theta\,\hat{i}_x + \sin\theta\,\hat{i}_y)$

$$\frac{d}{dt}(\hat{i}_r) = \boxed{-\sin\theta\frac{d\theta}{dt}}\,\hat{i}_x + \boxed{\cos\theta\frac{d\theta}{dt}}\,\hat{i}_y$$

chain Rule!

$$\frac{d}{dt}(\hat{i}_r) = \frac{d\theta}{dt}(-\sin\theta\,\hat{i}_x + \cos\theta\,\hat{i}_y)$$

$$\boxed{\frac{d}{dt}(\hat{i}_r) = \frac{d\theta}{dt}\,\hat{i}_\theta}$$

Need to prove on exam.

(b) $$\frac{d}{dt}(\hat{i}_\theta) = \frac{d}{dt}(-\sin\theta\,\hat{i}_x + \cos\theta\,\hat{i}_y)$$

$$\frac{d}{dt}(\hat{i}_\theta) = -\cos\theta\,\frac{d\theta}{dt}\,\hat{i}_x - \sin\theta\,\frac{d\theta}{dt}\,\hat{i}_y$$

$$\boxed{\frac{d}{dt}(\hat{i}_\theta) = -\frac{d\theta}{dt}\,\hat{i}_r}$$

Need to prove on exam.

$$\Rightarrow \vec{v} = \frac{d}{dt}(r\,\hat{i}_r) = \hat{i}_r\frac{dr}{dt} + r\,\frac{d\hat{i}_r}{dt}$$

$$\boxed{\vec{v} = \frac{dr}{dt}\,\hat{i}_r + r\,\frac{d\theta}{dt}\,\hat{i}_\theta}$$

Then for acceleration we find:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left[\frac{dr}{dt}\,\hat{i}_r + r\,\frac{d\theta}{dt}\,\hat{i}_\theta\right]$$

$$= \frac{d^2r}{dt^2}\,\hat{i}_r + \underline{\frac{dr}{dt}\,\frac{d\theta}{dt}\,\hat{i}_\theta} + \underline{\frac{dr}{dt}\,\frac{d\theta}{dt}\,\hat{i}_\theta} + r\,\frac{d^2\theta}{dt^2}\,\hat{i}_\theta$$

$$+ r\,\frac{d\theta}{dt}\left(-\frac{d\theta}{dt}\,\hat{i}_r\right)$$

$$\boxed{= \left(\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \right) \hat{i}_r + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} \right) \hat{i}_\theta}$$

How should we interpret these expressions for $\vec{v}$ and $\vec{a}$?

$$\vec{v} = \underbrace{\frac{dr}{dt}}_{\text{"radial velocity"}} \hat{i}_r + \underbrace{r\frac{d\theta}{dt}}_{\text{"tangential velocity"}} \hat{i}_\theta$$

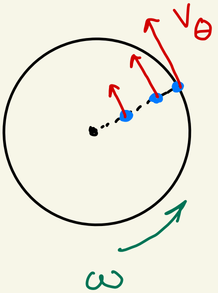
"radial velocity" (how fast moving away from origin)

"tangential velocity" (how fast moving around the origin)

The quantity $\boxed{\frac{d\theta}{dt} \equiv \omega}$ has a special name: "angular velocity". How fast object rotates.

⊛ Note that both radial and tangential velocities are actual velocities (they have units $\frac{m}{s}$), but angular velocity has different units: $\frac{1}{s}$ or revolutions/sec.

<u>Example</u>: Consider a solid wheel spinning at a constant speed, where the origin is taken to be the center.



(*) The <u>angular velocity</u> $\omega = \frac{d\theta}{dt}$ is constant. All points on the wheel complete 1 revolution in the same amount of time.

(*) The <u>tangential velocity</u> $v_\theta = r\frac{d\theta}{dt} = r\omega$ is larger farther from the center.

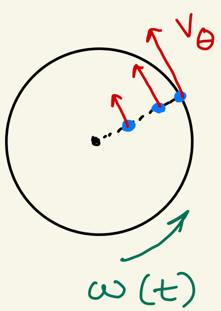
(*) The <u>radial velocity</u> for every point on the wheel is $v_r = \frac{dr}{dt} = 0$.

For acceleration we have 4 terms

$$\vec{a} = \Big(\frac{d^2r}{dt^2} \underbrace{- r\left(\frac{d\theta}{dt}\right)^2}_{\text{centripetal acceleration}}\Big)\hat{\imath}_r + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\hat{\imath}_\theta$$

(*) The quantity $\frac{d^2\theta}{dt^2}$ has a special name: "angular acceleration" $\boxed{\alpha \equiv \frac{d^2\theta}{dt^2}}$. Note that it is not a true acceleration but instead has units rev/sec^2.

→ It measures how the angular velocity $\omega = \frac{d\theta}{dt}$ is changing.



If $\omega(t)$ increases with time, then $\alpha > 0$.

(*) Centripetal acceleration $\vec{a}_c = -r\omega^2 \hat{\imath}_r$ will be especially important.

Example: We have introduced a lot of new terminology. Let's consider how all of these quantities behave for Earth rotating around the Sun.

Earth rotates at a nearly constant distance $r_E = 150 \times 10^6$ Km around the sun once per year. Find $\vec{v}, \vec{a}, \omega, \alpha$.

<u>Solution</u>: We know that $r(t) = 150 \times 10^6$ Km and $\omega(t) = \frac{2\pi}{365 \text{ days}}$, both of which are constant.

$$\Rightarrow \vec{v}(t) = \frac{dr}{dt}\hat{i}_r + r\omega\,\hat{i}_\theta$$

$$= 0\,\hat{i}_r + (1.5 \times 10^8 \text{ Km})\left(\frac{2\pi}{365 \text{ days}}\right)$$

$$\approx \boxed{30{,}000 \text{ m/s } \hat{i}_\theta}$$

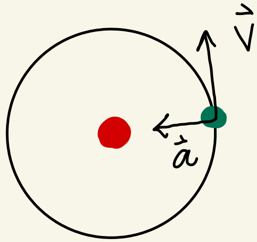
$$\vec{a}(t) = \Big(\underbrace{\cancel{\frac{d^2r}{dt^2}}}_{0} - r\omega^2\Big)\hat{i}_r + \Big(\underbrace{\cancel{r\alpha}}_{0} + \overbrace{\cancel{2\omega\frac{dr}{dt}}}^{0}\Big)\hat{i}_\theta$$

$$\boxed{\alpha = \frac{d\omega}{dt} = 0}$$

$$= -r\omega^2\,\hat{i}_r = -1.5 \times 10^{11}\,\text{m}\left(\frac{2\pi}{365 \text{ days}}\right)^2 \hat{i}_r$$

$$\boxed{= -0.006 \text{ m/s}^2\,\hat{i}_r}$$ (unnoticeably small).

(*) Interestingly, the acceleration is always radially inward:



This is true for all uniform ($\omega = $ const.) circular ($r = $ const.) motion.

But acceleration can point in other directions for any nonuniform or noncircular motion.

8.4 Using motion to find forces

Key Concept: If you know the motion of an object $r(t)$ and $\omega(t)$, you can find the total force acting on the object.

This follows trivially from $\vec{F} = m\vec{a}$:

$$r(t) \quad \text{and} \quad \omega(t) = \frac{d\theta}{dt}$$

$$\frac{dr}{dt} \rightarrow \frac{d^2r}{dt^2} \qquad \frac{d\omega}{dt}$$

$$\vec{a}(t) = \left(\frac{d^2r}{dt^2} - r\omega^2\right)\hat{i}_r + \left(r\frac{d\omega}{dt} + 2\frac{dr}{dt}\omega\right)\hat{i}_\theta$$

Once you find $\vec{a}$, you can easily find what must be the total force:

$$\boxed{\vec{F}_{total} = m\vec{a}}$$

Example: Exam 3 (2012) Q2, first part.

A flat circular curve, radius R, in the road says "Maximum speed v_{max}". If a car travels at v_{max} without slipping, what must be the coefficient of friction μ ?

Solution

⊛ Notice that we are given the exact motion of the object:

tangential velocity

$$r(t) = R \quad \text{and} \quad v_t = r\frac{d\theta}{dt} = v_{max}$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{v_{max}}{R}$$

Therefore, we should be able to find $\vec{F}_{total}$.

⊛ But what physical forces produce $\vec{F}_{total}$?

→ Only friction! Think about it... if the surface had no friction, could the car make the turn? No, it would just slide forward.

$$\Rightarrow F_{total} = f_\mu = \mu N = \mu mg$$

⊛ This is very common: motion gives you $\vec{F}_{total}$ but you have to relate it to the underlying physical forces.

Now we have

$$\vec{a} = \left(\cancel{\frac{d^2r}{dt^2}}^{\,0} - r\omega^2\right)\hat{i}_r + \left(r\cancel{\frac{d\omega}{dt}}^{\,0} + 2\cancel{\frac{dr}{dt}}^{\,0}\omega\right)\hat{i}_\theta$$

Since $r(t)$ and $\omega(t)$ are constant, three terms above vanish:

$$\vec{a} = -r\omega^2 = -R\left(\frac{V_{max}}{R}\right)^2$$

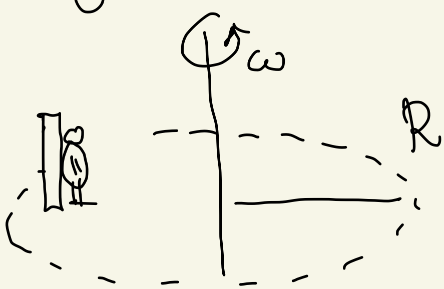
But $\vec{a} = \frac{1}{m}\vec{F}$

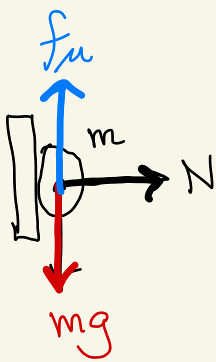
$$-\frac{V_{max}^2}{R} = -\frac{1}{m}(\underbrace{\mu mg}_{\text{friction}})$$

$$\Rightarrow \mu = \frac{V_{max}^2}{gR}$$

Example: In an amusement park ride, people stand with their backs against a circular wall that begins to spin along with the platform on which people stand. If the ride has radius R and the coefficient of friction between people and wall is μ, what angular velocity is needed so that the riders won't fall if the floor is removed?

Solution: Let's start by drawing a freebody diagram





(*) The only reason a rider doesn't fall off is due to friction counteracting gravity.

Larger $\omega \rightarrow$ larger $N \rightarrow$ larger f_μ.

For uniform circular motion, we have

$$\vec{a} = \left(\cancel{\frac{d^2r}{dt^2}} - r\omega^2\right)\hat{i}_r + \left(r\cancel{\frac{d\omega}{dt}} + 2\cancel{\frac{dr}{dt}}\frac{d\omega}{dt}\right)\hat{i}_\theta$$

$$\quad 0 \qquad\qquad\qquad\qquad 0 \qquad\quad 0$$

$$= -R\omega^2\,\hat{i}_r .$$

But $\vec{F} = m\vec{a} \Rightarrow \vec{F} = -mR\omega^2\,\hat{i}_r$.

Note that the centripetal force $\vec{F} = -mR\omega^2\,\hat{i}_r$ is not a new force. Instead it must result from some physical force ... in this case it is the normal force N.

$$\Rightarrow N = mR\omega^2$$

The minimum value of ω would lead person to just start slipping:

sliding friction $f_\mu = \mu N = \mu m R\omega^2$

Friction must exactly balance gravity

$$\Rightarrow f_\mu = mg$$

$$\Rightarrow \mu m R \omega^2 = mg$$

$$\Rightarrow \omega = \sqrt{\frac{g}{\mu R}}$$

(*) Note that this is independent of mass.

Just don't grease up your jacket before getting on ride!

In general it is difficult to integrate $\vec{v}(t)$ to obtain $\vec{r}(t)$ in polar coordinates (likewise, solving $\vec{v}(t) = \int a(t)\,dt$ is difficult).

This is because the basis vectors in general depend on time.

But there are two useful exceptions:

(a) Purely circular (Constant $r(t)$)

(b) Purely radial (Constant $\theta(t)$)

For (a) we can use

(1) $\alpha(t) = \dfrac{d\omega}{dt} \iff \omega(t) = \int \alpha(t)\,dt + C$

(2) $\omega(t) = \dfrac{d\theta}{dt} \iff \theta(t) = \int \omega(t)\,dt + C$

(3)
$$\left.\begin{aligned} \omega(t) &= \alpha t + \omega_0 \\ \theta(t) &= \tfrac{1}{2}\alpha t^2 + \omega_0 t + \theta_0 \\ \omega^2(t) &= \omega_0^2 + 2\alpha\theta \end{aligned}\right\} \text{Only for constant } \alpha.$$

(*) Analogous to previous equations for $a(t)$, $v(t)$, $x(t)$

For (b) we can use everything we've learned in the case of 1D motion, including

$$W = \int \vec{F}(r) \cdot d\vec{r} = \int \vec{F}(r) \cdot \left[dr\, \hat{i}_r + r\, d\theta\, \hat{i}_\theta \right]$$

We got this from $\vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt} \hat{i}_r + r \frac{d\theta}{dt} \hat{i}_\theta$

$$\Rightarrow d\vec{r} = dr\, \hat{i}_r + r\, d\theta\, \hat{i}_\theta$$

For $\vec{F}(r) = F(r)\hat{i}_r$, we find

$$W = \int F(r)\, dr = \Delta KE = -\Delta U$$

(*) Any purely radial force $F(r)\hat{i}_r$ has an associated potential energy function

$$\boxed{U(r) = -\int F(r)\, dr}$$