PHYS 206 Lecture 6

6.1) Position-dependent forces (springs, gravity, electric)
6.2) Conservative forces
6.3) Potential energy and energy conservation

6.1) Position-dependent forces

(*) One of the main uses of the work-energy theorem is that it allows us to analyze position-dependent forces

(*) Such forces were neglected in previous chapters because Newton's 2nd Law leads to differential equations:

$$\vec{F}(x) = m\frac{d^2\vec{r}}{dt^2} \quad \text{vs.} \quad \underbrace{\vec{F}(t) = m\frac{d^2\vec{r}}{dt^2} \Rightarrow \vec{a}(t) = \frac{\vec{F}(t)}{m}}_{\text{Can easily integrate to obtain } \vec{v}(t) \text{ and } \vec{r}(t).}$$

(*) The work-energy theorem doesn't give us all information, only partial information:

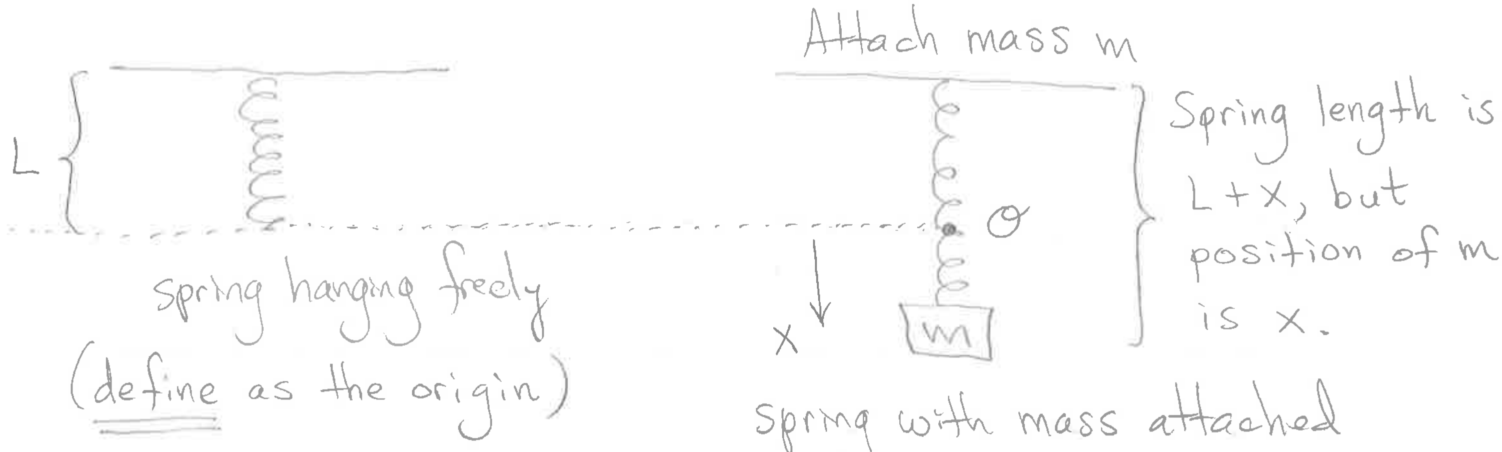
$$\boxed{v(\vec{r}) = \text{velocity as a function of position}}$$

instead of time

$$KE(\vec{r}_2) - KE(\vec{r}_1) = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}_{tot} \cdot d\vec{r}$$

$$\frac{1}{2}mv(\vec{r}_2) = \frac{1}{2}mv(\vec{r}_1) + \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}_{tot} \cdot d\vec{r}$$

* Spring force is a common example of a position-dependent force:



* Note: the force exerted by the spring must change with its stretch distance x because there is precisely one distance x_0 at which $F_s = mg$.

* Experiments show that many springs obey

Spring Force $\boxed{\vec{F}_s(\vec{x}) = -K\vec{x}}$ force is opposite push or pull direction. "spring constant" K

(as long as it isn't stretched extremely far)

* NOTE: "$\vec{x}$" is always the position (displacement) measured from the spring's resting length L.

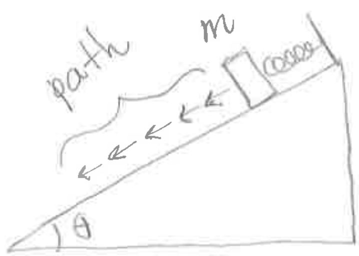
* Also, "$\vec{x}$" can be negative ⇒ force in opposite direction

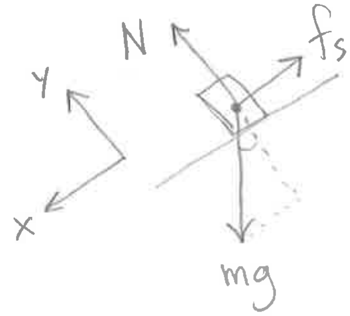
Example 1 : A spring with spring constant K is attached to the top of an inclined plane that makes an angle θ. If a block with mass m is attached to the spring and allowed to slide down the frictionless ramp, how much work will the spring do before the block comes to rest?

(Solution) In order to calculate work, we need two things:

(a) the force $\longrightarrow$ $F(x) = -Kx$

(b) the path $\longrightarrow$ we know block slides down... but how far??
(this is needed before we can compute work!)





The spring will stretch until block comes to rest.
Does that happen when $\vec{F}_{total} = 0$? No! Let's proove:

Y: $N - mg\cos\theta = 0$

X: $mg\sin\theta - Kx_f = 0 \Rightarrow \boxed{x_f = \frac{mg}{K}\sin\theta}$

Setting force equal to 0 at endpoint.

Now we know the path: $\int_0^{x_f} dx$

$$\Rightarrow W_{friction} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot d\vec{r} = \int_0^{x_f} (-Kx\,\hat{i}_x)\cdot(dx\,\hat{i}_x + dy\,\hat{i}_y)$$

$$= \int_0^{x_f} -Kx\,dx = -K\left[\frac{1}{2}x^2\right]_0^{x_f}$$

$$= -\frac{1}{2}K(x_f)^2 = -\frac{1}{2}K\left(\frac{mg}{K}\sin\theta\right)^2$$

$$\boxed{= -\frac{m^2g^2}{2K}\sin^2\theta}$$ Spring does negative work as expected ($\vec{F}_s$ opposite path)

Question: What was the block's change in kinetic energy?

For this we must not forget that gravity also does work.

Note that gravity will do positive work ($\vec{F}_g$ along the path)

Let's explicitly check work done by gravity:

$$W_{gravity} = \int_0^{x_f} (mg\sin\theta)\,dx = \frac{m^2g^2\sin^2\theta}{K}$$

This is twice the work done by friction.

Why aren't the two equal?

At $x = x_f = \frac{mg}{k}\sin\theta$ the forces are indeed all balanced,

but the block is still moving!

$$\frac{1}{2}mv_f^2 = \frac{m^2g^2}{k}\sin^2\theta - \frac{m^2g^2}{2k}\sin^2\theta = \frac{m^2g^2}{2k}\sin^2\theta$$

$$v(x_f) = g\sin\theta\sqrt{\frac{m}{k}}$$

Correct solution to problem: $\boxed{\text{enforce } v_f = 0 \Rightarrow W_{tot} = \Delta KE = 0}$

Block comes to rest only after the work done by gravity is equal and opposite to work done by spring force

$$W_{gravity} = \int_0^{x_c} mg\sin\theta\, dx = mg\sin\theta\, x_c$$

$$W_{spring} = \int_0^{x_c} -kx\, dx = -\frac{1}{2}kx_c^2$$

Set equal to each other.

$$\Rightarrow x_c = \frac{2mg\sin\theta}{k}$$ (correct stopping distance)

$$\Rightarrow W_{spring} = -\frac{1}{2}k\left[\frac{4m^2g^2\sin^2\theta}{k^2}\right] = -\frac{2m^2g^2\sin^2\theta}{k}$$

Gravitational force

⊛ It is important to distinguish gravitational force near Earth's surface ($f_g = mg$) and the correct gravitational force law at astronomical scales:

$$F_G = -G\frac{m_1 m_2}{r^2}$$

m_E = mass of earth = 5.97×10^{24} Kg

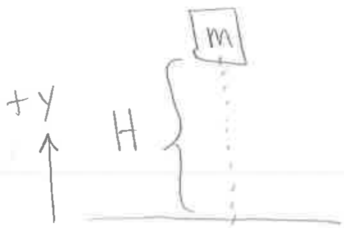
r_E = radius of earth = 6370 Km

even if you climb a mountain, your change from this value is a roundoff error.

G = gravitational constant = $6.67 \times 10^{-11}\ m^3/Kg/s^2$

$$\frac{G m_E}{r_E^2} = 9.81\ \frac{m}{s^2} = g$$

⊛ For objects near the surface, f_g is constant.



Work done by gravity as the block is dropped from rest:

$$W_g = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot d\vec{r} = \int_H^0 (-mg\,\hat{i}_y)\cdot(dx\,\hat{i}_x + dy\,\hat{i}_y)$$

$$= \int_H^0 -mg\,dy = -mgy\Big|_H^0 = mgH$$

Would the work done by gravity along this path change if the object had an initial velocity v_o?

Answer: No → the work done by gravity would be the same along that same path regardless of the block's velocity.

Question: What if we had chosen a coordinate system

$$W = \int_0^H \overset{+mg\hat{j}}{\vec{F}} \cdot \overset{dx\hat{i}+dy\hat{j}}{d\vec{r}} = \int_0^H mg\, dy$$

$$= mg \int_0^H dy = mgH$$

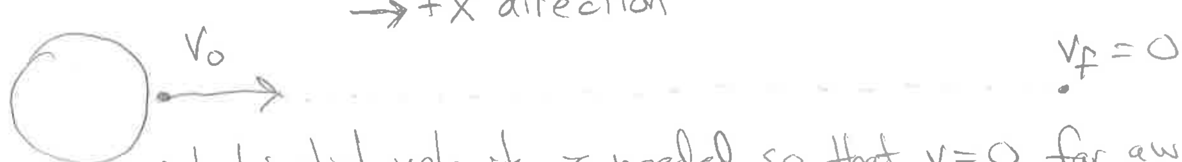
⊛ Regardless of coordinate system, the work done by a given force along a given physical path is the same.

⊛ For astronomical bodies

$$F_G = -G \frac{m_1 m_2}{r^2}$$

Example: Escape velocity of an object on Earth?

→ +x direction



What initial velocity is needed so that $v_f = 0$ far away?

Work-energy theorem

$$W_G = \Delta KE = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_0^2 = -\frac{1}{2}mv_0^2$$

$$W_G = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot d\vec{r} = \int_{r_E}^{\infty}\left(-G\frac{m_E m}{x^2}\hat{\imath}_x\right)\cdot\left(dx\,\hat{\imath}_x + dy\,\hat{\imath}_y\right)$$

$$= \int_{r_E}^{\infty} -G\frac{m_E m}{x^2}\,dx$$

$$= -Gm_E m\left[\frac{-1}{x}\right]_{r_E}^{\infty}$$

$$= -Gm_E m\left[-\frac{1}{\infty} + \frac{1}{r_E}\right]$$

$$= \frac{-Gm_E m}{r_E}$$

Set $-\frac{1}{2}\cancel{m}v_0^2 = -\frac{Gm_E\cancel{m}}{r_E}$ (independent of object's mass m)

$$\Rightarrow v_0^2 = \frac{2GM_E}{r_E}$$

$$\Rightarrow v_0 = 11200\ \frac{m}{s}$$

5 steps for solving work-energy questions (x-dependent forces)

① Choose a coordinate system:

(a) Origin (for springs, origin always at natural rest length)

(b) $+x$ and $+y$ directions

② Write all forces in vector notation based on your choice of coordinate system.

Example:

$$\vec{F}(x) = -F(x)\,\hat{i}_x .$$

③ Determine start x_o and end x_f of path along which the object moves.

④ Evaluate $\Delta KE = W_{total}$

$$\frac{1}{2}mv^2(x_f) - \frac{1}{2}mv^2(x_i) = \int_{x_o}^{x_f} \vec{F}\cdot d\vec{r}$$

⑤ After taking dot product of $\vec{F}$ in vector notation (from step ② above) and $d\vec{r} = dx\,\hat{i}_x + dy\,\hat{i}_y$, you will end up with just a one-dimensional integral.

Remember $\hat{i}_x \cdot \hat{i}_x = 1$

$\hat{i}_x \cdot \hat{i}_y = 0$

$\hat{i}_y \cdot \hat{i}_y = 1$

Both gravity and the electrostatic force are examples of "conservative forces" $\longrightarrow$ forces for which the total mechanical energy of an object is independent of time.

6.2) Conservative forces

⊛ We have already briefly discussed gravitational potential energy (near Earth's surface)

Lift a block of mass m and it now has the potential to gain kinetic energy.

How much kinetic energy can it gain?

Answer: From work-kinetic energy theorem we find

$$W_g = KE_f - \cancel{KE_i}^{\,0} \quad \text{(drop from rest)}$$

$$\Rightarrow KE_f = W_g = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}_g \cdot d\vec{r} = \int_h^0 (-mg\hat{\imath}_y)\cdot(dx\,\hat{\imath}_x + dy\,\hat{\imath}_y)$$

$$= \int_h^0 -mg\,dy = -mg\left[y\right]_h^0 = mgh$$

Define as the gravitational potential energy.

As the block falls, it gains Kinetic energy and loses the potential to gain more energy (in equal amounts)

$\Rightarrow \quad \Delta KE = -\Delta PE$ (we usually use "U" for potential energy)

$$\Rightarrow \quad \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}_g \cdot d\vec{r} = -\Delta U$$

$$\boxed{U(\vec{r}_2) - U(\vec{r}_1) \equiv -\int_{\vec{r}_1}^{\vec{r}_2} \vec{F}_g \cdot d\vec{r}}$$

(*) This is the definition of potential energy

Some Very Important Points :

(a) Notice we haven't uniquely defined $U(\vec{r})$ ... we have only defined a change in U.

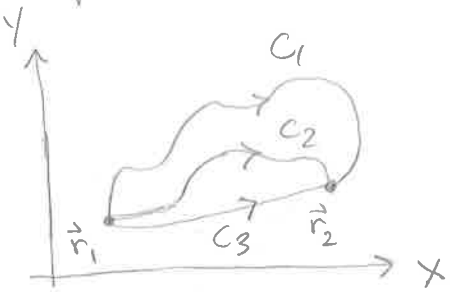
(*) Only changes in the potential energy have a physical meaning. We can choose $U = 0$ anywhere we want (but once it is chosen, it cannot be changed for the rest of the problem)

(b) The definition above only makes sense if the quantity

$\int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r}$ is independent of the actual path between

$\vec{r}_1$ and $\vec{r}_2$ (otherwise $U(\vec{r})$ is ill-defined)

Example:



(overhead view)

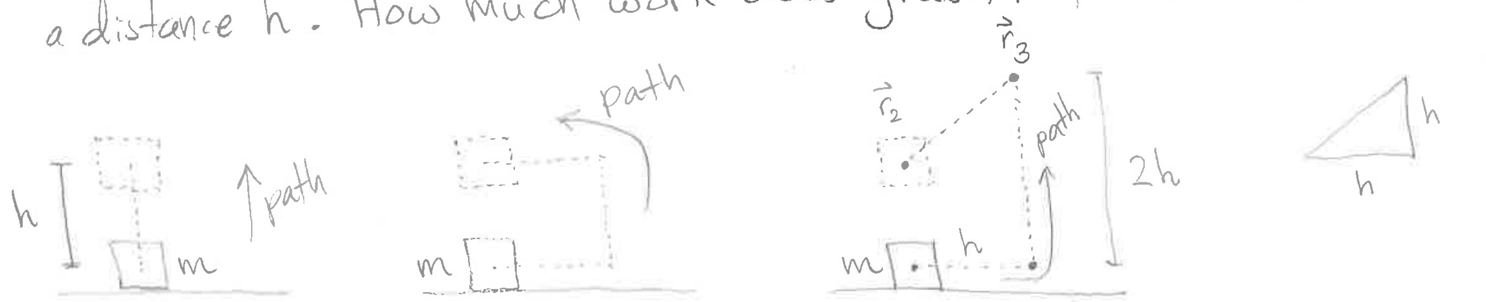
Three different paths must all yield identical values for $\int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r}$.

⊛ Friction is not conservative:

Work done by friction depends on path:

$$\left| \int_{C_1} \vec{F}_f \cdot d\vec{r}_1 \right| > \left| \int_{C_2} \vec{F}_f \cdot d\vec{r} \right| > \left| \int_{C_3} \vec{F}_f \cdot d\vec{r} \right|$$

Example: let's look at three paths in moving a block upward a distance h. How much work does gravity perform?



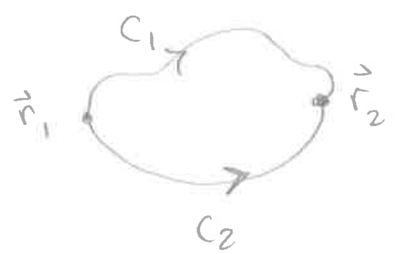
$$W_g = \int \vec{F}_g \cdot d\vec{r}$$
$$= -mgh$$

$$W_g = \int \vec{F}_g \cdot d\vec{r}$$
$$= 0 - mgh + 0$$

Path independent!

$$W_g = \int \vec{F}_g \cdot d\vec{r}$$
$$= 0 - 2mgh + \int_{\vec{r}_3}^{\vec{r}_2} \vec{F} \cdot d\vec{r}$$
$$= -2mgh + \int_0^{\sqrt{2}h} mg \cos 45^\circ \, dr$$
$$= -2mgh + mg \frac{\sqrt{2}}{2} \sqrt{2} h = -mgh$$

(*) Path independence also implies that the work done along a <u>closed</u> path is 0.



$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r} = -\int_{-C_2} \vec{F}\cdot d\vec{r}$$

$$\Rightarrow \int_{C_1 - C_2} \vec{F}\cdot d\vec{r} = 0$$

(*) <u>Definition</u> of "conservative force": a force for which the work done from $\vec{r}_1$ to $\vec{r}_2$ is independent of path

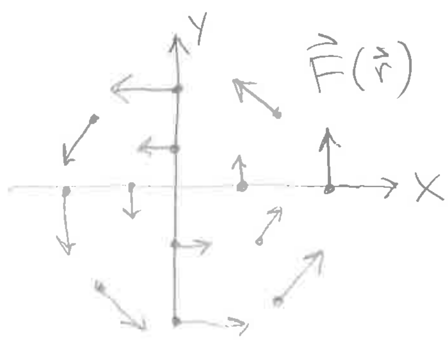
(c) If we move an object <u>against</u> a conservative force, the potential energy <u>increases</u>.

(Likewise if an object moves in the direction of a conservative force, then the potential energy <u>decreases</u>)

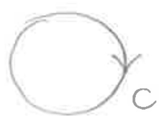
Since we have moved against the force, that force has the potential to increase the kinetic energy if we now release the object.

⊛ In general it is not easy to prove that a given force does work that is path independent

Example: Is the force vector field shown conservative?



No! A circular clockwise path would result in negative work since $d\vec{r}$ would always be opposite the force direction



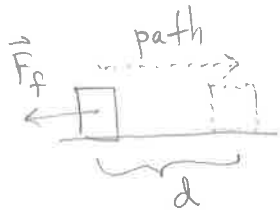
$$\int_C \vec{F}\cdot d\vec{r} < 0.$$

⊛ For now, any one-dimensional force depending only on x and pointing in ±x direction ($\vec{F}(x) = F(x)\hat{\imath}_x$) is conservative:

$$\int_{\vec{r}_1}^{\vec{r}_2} \vec{F}(\vec{r})\cdot d\vec{r} = \int_{\vec{r}_1}^{\vec{r}_2} (F(x)\hat{\imath}_x)\cdot(dx\,\hat{\imath}_x + dy\,\hat{\imath}_y)$$

$$= \int_{x_1}^{x_2} F(x)\,dx = f(x_2) - f(x_1)$$ antiderivative function f.

Wait!! Isn't this just what we did for friction??



$$W_f = \int_0^d F_f\,dx = \int_0^d -\mu mg\,dx = -\mu mgd$$

Subtle point: No, the friction force is not a function of x!

The friction force has a different sign depending on the direction of its velocity.

If I tell you the location of the block, you cannot tell me the friction force

(a) $F_f = 0$ if block is at rest

(b) $F_f = -\mu N$ if velocity is to the right

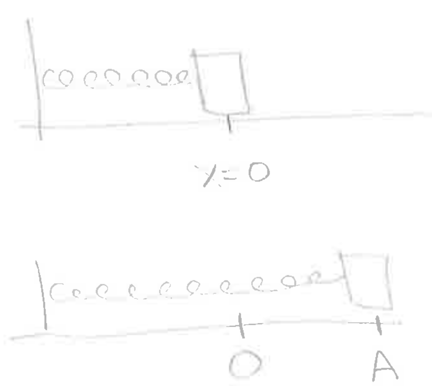
(c) $F_f = \mu N$ if velocity is to the left.

⊛ Only if we can uniquely define a force knowing only its position $\vec{F}(x) = F(x)\,\hat{i}_x$ is the force conservative.

6.3) Potential energy functions and energy conservation

Example: Spring force $F(x) = -Kx$.

$$W = \int_a^b F(x)\,dx = \int_a^b -Kx = \left[-\frac{1}{2}Kx^2\right]_a^b$$



If I stretch spring a distance A, I know exactly what force the spring exerts on the block, regardless of its velocity or any other features of its motion.

⊛ Spring potential energy

$$U(b)-U(a) = -\int_a^b \vec{F}\cdot d\vec{r}$$

$$U(b)-U(a) = -\int_a^b (-kx\hat{\imath}_x)\cdot(dx\,\hat{\imath}_x + dy\,\hat{\imath}_y)$$

$$U(b)-U(a) = \int_a^b kx\,dx = \left[\tfrac{1}{2}kx^2\right]_a^b$$

"Potential energy function"

$$U(b)-U(a) = \tfrac{1}{2}kb^2 - \tfrac{1}{2}ka^2$$

"Potential energy difference"

Free to choose location where $U=0$

$$\rightarrow U(x=0)=0$$

$$\Rightarrow U(x)-U(0) = \tfrac{1}{2}k\left[x^2-0^2\right]$$

$$\boxed{U_s(x) = \tfrac{1}{2}kx^2}$$

Spring potential energy function.

⊛ For any arbitrary force depending on only x:

$$U_F(x) = -\int F(x)\,dx$$

Potential energy function is <u>negative</u> antiderivative of $F(x)$

(*) For a 2D force $\vec{F}(\vec{r}) = f(x)\,\hat{i}_x + g(y)\,\hat{i}_y$,

$$\mathcal{U}(x,y) = -\int \vec{F}\cdot d\vec{r} = -\int \left(f(x)\,\hat{i}_x + g(y)\,\hat{i}_y\right)\cdot\left(dx\,\hat{i}_x + dy\,\hat{i}_y\right)$$

$$= -\int_{x_1}^{x_2} f(x)\,dx - \int_{y_1}^{y_2} g(y)\,dy$$

$$= \mathcal{U}_x(x) + \mathcal{U}_y(y)$$

Just treat the x and y separately and compute the individual potential functions

(*) Suppose we are given instead the potential function $\mathcal{U}(x)$ and asked to find the force:

$$\boxed{F(x) = -\frac{d\mathcal{U}}{dx}}$$

DON'T FORGET THE MINUS SIGN !!

(*) Very <u>easy</u> to get force from $\mathcal{U}$:

$$\mathcal{U}(x) = \frac{-1}{x} \Rightarrow F(x) = \frac{-1}{x^2}$$

* Conservation of Mechanical Energy

Recall: $W = \int_a^b \vec{F}_{tot} \cdot d\vec{r} = \Delta KE$ and

$$U(b) - U(a) = -\int_a^b \vec{F} \cdot d\vec{r}$$

$\Rightarrow$ $U(b) - U(a) = -\Delta KE$ (when only conservative forces are present)

$$\Rightarrow U(b) - U(a) = KE(a) - KE(b)$$

$$\Rightarrow \underbrace{KE(a) + U(a)}_{\substack{\text{Mechanical} \\ \text{energy at } x=a}} = \underbrace{KE(b) + U(b)}_{\substack{\text{Mechanical} \\ \text{energy at } x=b}}$$

* For conservative forces, the mechanical energy $KE + U$ is constant for all x ... we say it is "conserved"

Example: Exam 2 (2013) Problem 2

The Kinetic energy of an object with mass m is measured to be $KE(x) = K_o\left(1 - \frac{x^2}{L^2}\right)$ within the region $0 \le x \le L$. Assuming only conservative forces act on the object, what is $F(x)$?

(Solution): The Key word to notice is "conservative".

$\Rightarrow$ Use Conservation of Energy:

$$KE(x) + U(x) = \text{constant}$$

Do we Know this constant? Nope, but so what?!

$$K_o\left(1 - \frac{x^2}{L^2}\right) + U(x) = C$$

$$U(x) = C - K_o\left(1 - \frac{x^2}{L^2}\right)$$

$$\Rightarrow F(x) = -\frac{dU}{dx}$$

drops out (fortunately!)

$$= -\frac{d}{dx}\left(C - K_o\left(1 - \frac{x^2}{L^2}\right)\right)$$

$$= -\frac{d}{dx}\left(K_o \frac{x^2}{L^2}\right)$$

$$F(x) = -\frac{2K_o}{L^2}x$$

⊛ A few more words on 2D functions

How can we compute the force from $U(x,y)$??

Start with

$$U(\vec{r}_2) - U(\vec{r}_1) = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r}$$

For a very small displacement $d\vec{r}$ we have

$$dU = -\vec{F} \cdot d\vec{r} = -(F_x \hat{i}_x + F_y \hat{i}_y) \cdot (dx\, \hat{i}_x + dy\, \hat{i}_y)$$

$$\Rightarrow dU = -F_x\, dx - F_y\, dy$$

Trick: We can extract F_x by moving directly in the x-direction (where $dy = 0$)

$$\Rightarrow dU = -F_x\, dx \Big|_{dy=0}$$

$$\Rightarrow F_x = -\frac{dU(x,y)}{dx}\Big|_{dy=0}$$

⊛ The quantity $\left.\frac{dU(x,y)}{dx}\right|_{dy=0}$ has a special name:

"partial derivative" $\frac{\partial U}{\partial x} \equiv \left.\frac{dU(x,y)}{dx}\right|_{y=\text{const.}}$

Likewise $\frac{\partial U}{\partial y} \equiv \left.\frac{dU(x,y)}{dy}\right|_{x=\text{const.}}$

⊛ Just treat like a regular derivative but treat the other variable as if it were a constant.

Example: Let $U(x,y) = 3x^2y + y^2x$.

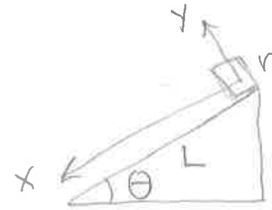
$$F_x = -\frac{\partial U}{\partial x} = -6xy - y^2 \quad (y \text{ just a constant})$$

$$F_y = -\frac{\partial U}{\partial y} = -3x^2 - 2xy \quad (x \text{ treated like a constant})$$

$$\Rightarrow \boxed{\vec{F}(x,y) = \left(-6xy - y^2\right)\hat{\imath}_x + \left(-3x^2 - 2xy\right)\hat{\imath}_y}$$

$$\vec{F}(1,1) = \left(-6\cdot 1\cdot 1 - 1^2\right)\hat{\imath}_x + \left(-3\cdot 1^2 - 2\cdot 1\cdot 1\right)\hat{\imath}_y$$

$$= -7\,\hat{\imath}_x - 5\,\hat{\imath}_y$$



Example: Exam 2 (2009) Problem 3

A block of mass m slides down an inclined plane. In addition to gravity there is another force whose potential energy function is $U(x,y) = \beta(x^2+y^2)$. Find this force and determine how fast the block moves when it reaches the bottom.

(Solution)

We can find the components of the force:

$$F_x = -\frac{\partial U}{\partial x} = -2\beta x$$

$$F_y = -\frac{\partial U}{\partial y} = -2\beta y$$

$$\Rightarrow \vec{F} = F_x \hat{i}_x + F_y \hat{i}_y$$

It is simplest to find the final velocity using energy conservation:

$$KE^i = 0$$
$$U_g^i = mgH = mgL\sin\theta$$
$$U_m^i = \beta(x^2+y^2) = \beta(0^2+0^2)$$

Total: $E_{tot}^i = mgL\sin\theta$

$$KE_f = \frac{1}{2}mv^2$$
$$U_g^f = 0$$
$$U_m^f = \beta(x^2+y^2) = \beta L^2$$

$$E_{tot}^f = \frac{1}{2}mv^2 + \beta L^2$$

$$\Rightarrow \quad mgL\sin\theta = \frac{1}{2}mv^2 + \beta L^2$$

$$\Rightarrow \quad \frac{1}{2}mv^2 = mgL\sin\theta - \beta L^2$$

$$v = \left(\frac{2}{m}\left[mgL\sin\theta - \beta L^2\right]\right)^{1/2}$$