PHYS 207 Electricity and Magnetism

5.) Applications of Gauss's Law

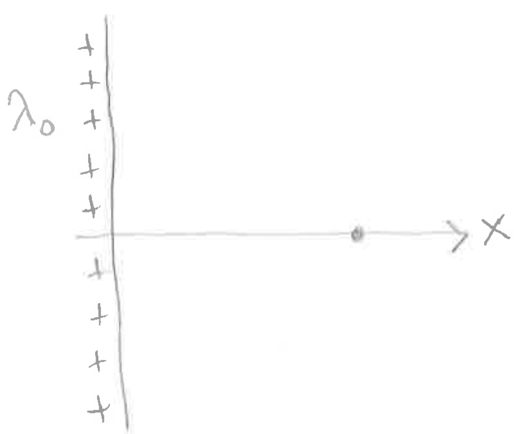
5.1) Electric field around symmetrical charge distributions

5.2) Properties of conductors

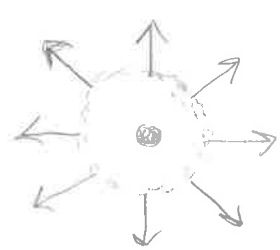
5.1) $\vec{E}$ field for symmetrical charge distributions

(*) From Gauss's Law we can easily compute $\vec{E}$ near conductors and insulators if they exhibit symmetry

Example: Consider the infinite line charge with uniform charge density λ_0 that we studied previously

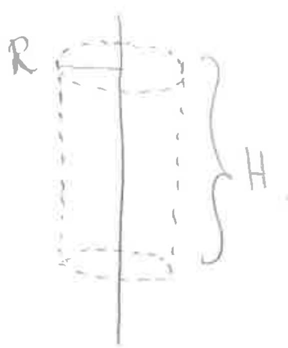


Note that by symmetry the electric field magnitude is constant along a cylinder:



(overhead view)

From Gauss's Law, the flux through a cylinder of radius R and height H is



$$\oint \vec{E}\cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0}$$

First, consider the left-hand side

$$\oint \vec{E}\cdot d\vec{A} = \oint (\vec{E}\cdot\hat{n})\, dA$$

1. $\hat{n} = \hat{\imath}_r$
2. $\vec{E}\cdot\hat{n} = (E(r)\hat{\imath}_r)\cdot\hat{\imath}_r = E = \text{constant}$

   From symmetry E points radially outward
3. $\vec{E}\cdot\hat{n}\big|_s = E(R)$
4. $dA \rightarrow$ small sections of cylinder
5. $\oint \vec{E}\cdot d\vec{A} = E(R)\oint dA = E(R)A = E(R)\left[2\pi R H\right]$

Note that for the top and bottom circular areas $\hat{n} = \hat{\imath}_z$ and $\vec{E}\cdot\hat{n} = 0$.

Therefore, $\oint \vec{E}\cdot d\vec{A} = 2\pi R H \cdot E(R)$

E(R) is currently <u>unknown</u>, but we will solve for it since we can find q_{enc}.

How much charge is inside the cylinder?

$$\lambda = \lambda_o \text{ and length} = H$$

$$\Rightarrow q_{enc} = \lambda_o H$$

Finally, from Gauss's Law we find

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_o}$$

$$2\pi R_\circ H \cdot E(R_\circ) = \frac{\lambda_o H}{\varepsilon_o}$$

$$\Rightarrow \boxed{E(R_\circ) = \frac{\lambda_o}{2\pi\varepsilon_o} \frac{1}{R_\circ}}$$ (exactly as we found before)

(*) Even though $\oint \vec{E} \cdot d\vec{A}$ looks more complicated than $\frac{q_{enc}}{\varepsilon_o}$, very often $\oint \vec{E} \cdot d\vec{A}$ will be <u>simple</u> and <u>q_{enc}</u> will be <u>difficult</u> to calculate.

<u>Example</u>: Suppose we have a non-uniformly charged spherical insulator where the volume charge density is $\rho(r) = \rho_o\left(1 - \frac{r}{R_\circ}\right)$, where $R_\circ$ is the sphere's radius. What is $\vec{E}$ inside and outside the sphere?

For $r<R$ we draw a spherical Gaussian surface inside the sphere.

From symmetry the electric field at any point on this imaginary surface is radially outward and has constant magnitude

Gauss's Law: $\oint \vec{E}\cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0}$

$\oint \vec{E}\cdot d\vec{A}$:
1. $\hat{n} = \hat{i}_r$
2. $\vec{E}\cdot\hat{n} = (E\,\hat{i}_r)\cdot\hat{i}_r = E = \text{constant}$
3. $\vec{E}\cdot\hat{n}\big|_S = E(r)$
4. dA small part of sphere
5. $\oint \vec{E}\cdot d\vec{A} = \int (\vec{E}\cdot\hat{n})\,dA = E(r)A = E(r)\cdot 4\pi r^2$

(*) Very important: surface area of Gaussian sphere is

$4\pi r^2$ not $4\pi R^2$

$4\pi R^2$ is the surface area of the physical sphere, but $r<R$.

How much charge is inside this Gaussian surface?

Recall: Linear charge density: $\lambda = \frac{dq}{dx} \Rightarrow \boxed{dq = \lambda\, dx}$

Surface charge density: $\sigma = \frac{dq}{dA} \Rightarrow \boxed{dq = \sigma\, dA}$

Volume charge density: $\rho = \frac{dq}{dV} \Rightarrow \boxed{dq = \rho\, dV}$

In this case $dq = \rho\, dV = \rho_0 \left(1 - \frac{r}{R}\right) dV$, but

what is dV ??

Trick: $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2 \Rightarrow \boxed{dV = 4\pi r^2\, dr}$

$$\Rightarrow q_{enc} = \int dq = \int_0^r \rho_0 \left(1 - \frac{r}{R}\right) 4\pi r^2\, dr$$

(*) Upper integration limit is r not R. The Gaussian surface contains only a part of the total charge inside R.

$$\Rightarrow q_{enc} = 4\pi \rho_0 \int_0^r \left(r^2 - \frac{r^3}{R}\right) dr$$

$$= 4\pi \rho_0 \left[\frac{1}{3} r^3 - \frac{1}{4}\frac{r^4}{R}\right]_0^r$$

$$= 4\pi \rho_0 r^3 \left[\frac{1}{3} - \frac{1}{4}\frac{r}{R}\right]$$

Finally, we find

$$\oint \vec{E} \cdot d\vec{A} = \frac{\rho_{enc}}{\varepsilon_0}$$

$$E\, 4\pi r^2 = \frac{1}{\varepsilon_0} 4\pi \rho r^3 \left[\frac{1}{3} - \frac{1}{4}\frac{r}{R} \right]$$

$$\Rightarrow \quad E = \frac{\rho_0}{\varepsilon_0} \left[\frac{r}{3} - \frac{r^2}{4R} \right]$$

(*) <u>Note</u>: It is crucial to write dV (or dA for surface charge distributions) in terms of <u>only</u> the variable that ρ depends on.

To do this, write the formula for V and differentiate.

<u>Example</u>: Long cylinder with charge density $\rho(r)$

$$\Rightarrow \quad V = (\pi r^2) H$$

$$\frac{dV}{dr} = 2\pi r H \Rightarrow \boxed{dV = 2\pi r H\, dr}$$

<u>Example</u>: Circular charge distribution with surface charge density $\sigma(r)$

$$\Rightarrow \quad A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r = \boxed{dA = 2\pi r\, dr}$$

5-step process for computing $\vec{E}$ using Gauss's Law

① To find $\vec{E}$ at some point p, draw a "hypothetical" surface that

(a) goes through the point p and

(b) takes advantage of the charge distribution's symmetry

( → spherical ,  → cylindrical,  → translational)

② Compute electric flux through the hypothetical surface.

Note that $\vec{E}\cdot\hat{n}$ should be a constant, so that

$$\oint \vec{E}\cdot d\vec{A} = E\oint dA = E\cdot(\text{Surface Area})$$

③ Compute charge inside hypothetical surface from

$$q_{enc} = \int dq \quad \begin{cases} dq = \lambda\, dl & \text{(line charges)} \\ dq = \sigma\, dA & \text{(surface charges)} \\ dq = \rho\, dV & \text{(volume charges)} \end{cases}$$

Note that $\lambda, \sigma,$ and ρ will be given for nonuniform charge distributions but must be computed for uniform distributions

④ Write dl, dA, dV in terms of the variable that λ, σ, ρ depends on

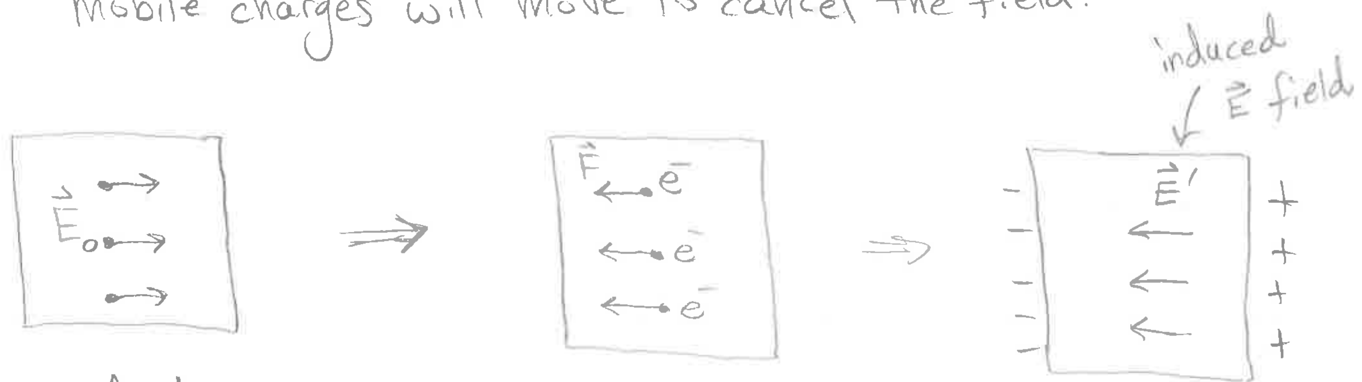
Example: $dV = 4\pi r^2 dr$ (sphere), $dV = 2\pi r h\, dr$ (cylinder), etc.

⑤ Integrate to get q_{enc} → then $E = \dfrac{1}{\text{surface area}} \dfrac{q_{enc}}{\varepsilon_0}$

5.2) Properties of Conductors

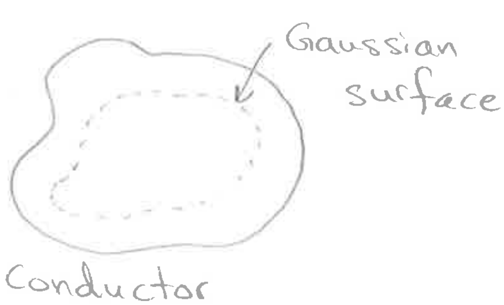
⊛ We can use Gauss's Law to derive some simple properties of conductors

⊛ Key Point: in a static system, the electric field inside a conductor must be identically 0. Otherwise the mobile charges will move to cancel the field.



Conductor with $\vec{E}_0$ field $\Rightarrow$ Electrons will move leftward $\Rightarrow$ New configuration produces opposite $\vec{E}$ field until $\boxed{\vec{E}_0 + \vec{E}' = 0}$

Example: Suppose an excess $+Q$ charge is placed on a conductor. How will the charge distribute itself inside the conductor?

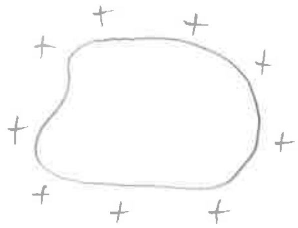


$$\oint \vec{E} \cdot d\vec{A} = 0 \quad \text{since } \vec{E} = 0$$

$$\Rightarrow q_{enc} = \oint \vec{E} \cdot d\vec{A} = 0$$

The charge enclosed $q_{enc} = 0$

$\Rightarrow$ All net charge is pushed to the surface



This configuration maximizes the distance between the repulsive charges.

Example: Suppose there is a hollow conducting sphere with a charge Q_1 in the inner nearly empty space and a net charge $-Q_2$ on the conductor itself. What is the net charge on the conductor's inner surface and outer surface?

Solution: physically what do we expect to happen? The positive charge inside will attract negative charge, but how much?

Gaussian surface

Draw Gaussian surface inside the conductor. Since $\vec{E} = 0$ inside, $\oint \vec{E} \cdot d\vec{A} = 0 = q_{enc}$.

Exactly $\boxed{-Q_1}$ charge distributes itself on inner surface

Since $-Q_1$ has moved to inner surface, there will be a net charge of $-Q_2 - (-Q_1) = \boxed{-Q_2 + Q_1}$ left on the outer surface.

Of couse $Q_{inner} + Q_{outer} = -Q_2$ (the total net charge on the conductor)