4. Gauss's Law

4.1) Electric flux

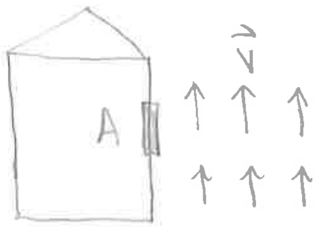
* Vector fields are ubiquitous in nature and in physics
  - → So far we have talked about
    - (a) velocity field of wind $\vec{v}(\vec{r})$
    - (b) force field $\vec{F}(\vec{r})$
    - (c) electric field $\vec{E}(\vec{r})$

* For any vector field we can define the "flux" of the field through a surface S.

  Qualitatively, the flux represents how much of the field "goes through" the surface.

Example: Consider wind blowing in the vicinity of an open window of area A.

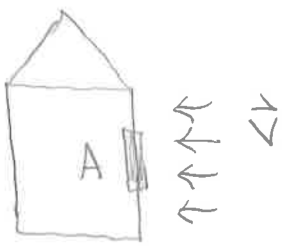
(Case 1): wind blows parallel to window



Since wind is parallel to window, no air flows through.

$\Rightarrow$ Flux = 0

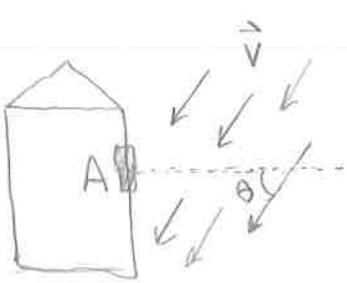
(Case 2): wind blows perpendicular to window



Since wind is perpendicular to wind, airflow is maximal.

$\Rightarrow$ Flux = Av gives the volume of air flowing in every second

(Case 3): wind blows at angle θ



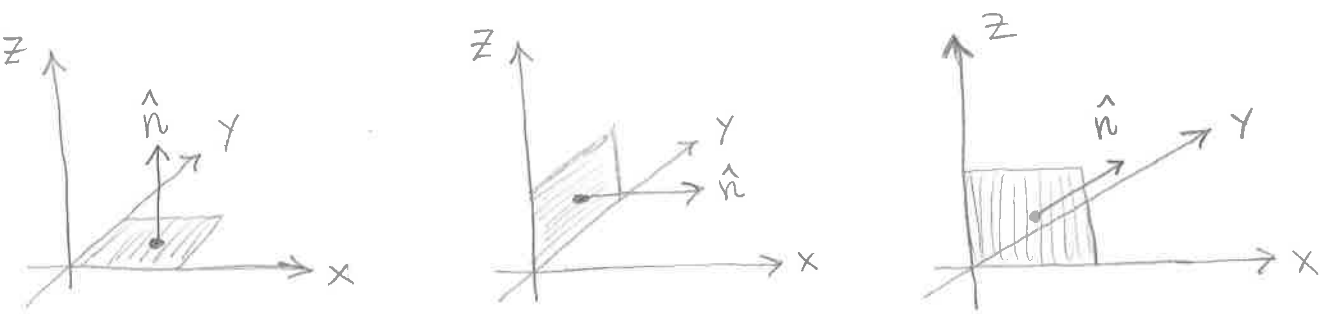
How do we compute how much air flows in? Only component of $\vec{v}$ perpendicular to window matters:

$\Rightarrow$ Flux = $Av\cos\theta$

(*) First crucial point : flux depends on angle between vector field and surface S.

How to formally define direction of a surface??

→ with the "normal vector" $\hat{n}$:



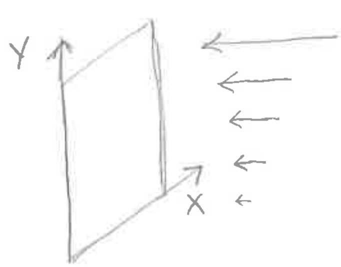
$\hat{n} = \pm \hat{\imath}_z$ $\quad$ $\hat{n} = \pm \hat{\imath}_x$ $\quad$ $\hat{n} = \pm \hat{\imath}_y$

Then for a constant vector field $\vec{v}$, the flux is

$$\boxed{\phi = \vec{v} \cdot \vec{A} = \vec{v} \cdot (A\hat{n})}$$

↑ Greek letter phi for flux

(*) Life is often not so simple, though. The vector field $\vec{v}$ can vary across the surface



$\vec{v}$ increases with y in this example

As you might have guessed, in this case we break up area $\vec{A}$ into small segments $d\vec{A}$ and integrate:

$$\boxed{\phi = \int \vec{v} \cdot d\vec{A}}$$

Most general definition.

(*) Particularly useful will be the concept of

$$\boxed{\text{"Electric Flux"} \equiv \phi_E \equiv \int \vec{E} \cdot d\vec{A}}$$

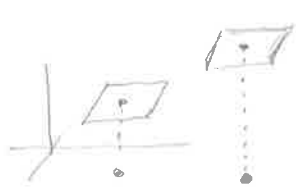
4.2) <u>Surface integrals</u>

The definition of electric flux is given in terms of what we call a "surface integral".

(*) The integration region is an area, such as $dx\,dy$, $dy\,dz$, or $dx\,dz$

How to compute surface integrals?

Note that we will need the surface's <u>orientation</u> and <u>location</u>.



Same orientation but different locations.

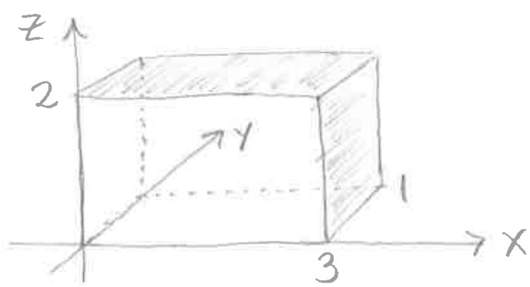
(*) 5-step process

(1) Determine the normal vector $\hat{n}$ to the surface (orientation)

(2) Write $\vec{E}$ in vector notation and evaluate $\vec{E}\cdot\hat{n}$ (this will often simplify things greatly and always results in a scalar quantity $f(x,y,z)$)

(3) Evaluate $\vec{E}\cdot\hat{n} = f(x,y,z)$ on the surface S (location)

Example:



depends only on x and y now

Top surface has $z=2 \Rightarrow f(x,y,z=2)$

Right surface has $x=3 \Rightarrow f(x=3,y,z)$

depends only on y and z now

(4) Determine dA and check if $\vec{E}\cdot\hat{n} = f(x,y,z)$ is constant along one of the two directions (hint: it will be!)

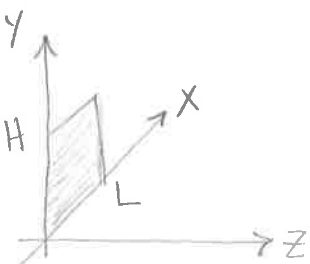
Top surface spans x and y directions $\rightarrow$ $dA = dx\,dy$

Right surface spans y and z directions $\rightarrow$ $dA = dy\,dz$

(5) Integrate (it should just end up as a one-dimensional integral)

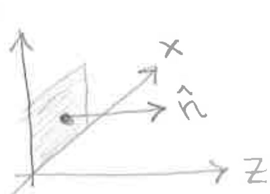
Example: Suppose the electric field in a region of space is given by $\vec{E}(x,y,z) = \underbrace{(a+by+cz)}_{\text{magnitude}}\ \underbrace{\hat{\imath}_z}_{\text{direction}}$.

What is the electric flux through the surface below?



Surface lies in x-y plane.

① Normal vector $\hat{n} = \hat{\imath}_z$



② $\vec{E}$ is already given to us in vector notation

$$\Rightarrow \vec{E} \cdot \hat{n} = (a+by+cz)\,\hat{\imath}_z \cdot \hat{\imath}_z$$

$$= a+by+cz$$

③ Evaluate $\vec{E} \cdot \hat{n} = a+by+cz$ on surface. Note that

$z = 0$ on surface:

$$\vec{E} \cdot \hat{n} = a+by+cz = a+by+c(0) = a+by$$

(4) Determine dA and check if $\vec{E}\cdot\hat{n}$ is independent of one of the integration variables.

Since the surface lies in the xy plane, $dA = dx\,dy$.

Note that $\vec{E}\cdot\hat{n}$ is independent of x.

(5) Integrate.

$$\phi_E = \int \vec{E}\cdot d\vec{A} = \int \vec{E}\cdot\hat{n}\,dA$$

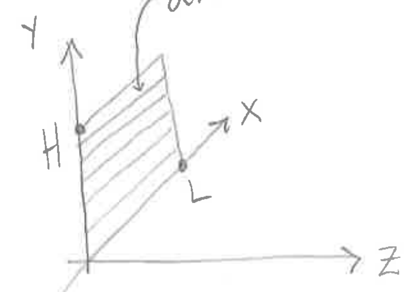
$$= \int (a+by)\,dx\,dy$$

$$= \int_0^H (a+by)\,dy \int_0^L dx$$

Alternatively

Since $\vec{E}\cdot\hat{n}$ is independent of x, we can write:

$dA = L\,dy$



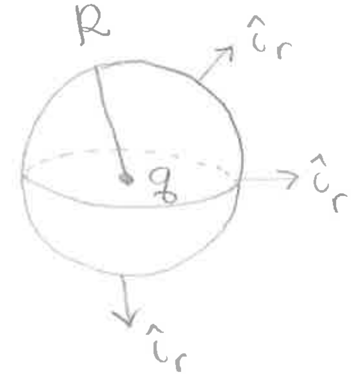
$$= L\left(ay + \tfrac{1}{2}by^2\right)\Big|_0^H$$

$$\boxed{\phi_E = L\left(aH + \tfrac{1}{2}bH^2\right)}$$

Electric flux will always be a scalar quantity

Example : Consider a point charge Q located at the origin. What is the electric flux through a sphere of radius R centered at the origin?

Solution: Follow 5-step process



(1) Unit normal vector $\hat{n}$?

The normal vector to the sphere is just $\hat{\imath}_r$.

(2) Write $\vec{E}$ in vector notation:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\imath}_r$$

(Note that we have "r" instead of "R" at this stage)

and evaluate

$$\vec{E}\cdot\hat{n} = \left(\frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\imath}_r\right)\cdot\hat{\imath}_r = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$$

(3) Evaluate $\vec{E}\cdot\hat{n}$ on the surface:

The surface is defined by $r = R$

$$\Rightarrow \vec{E}\cdot\hat{n} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R^2}$$

(4) Determine dA and check whether $\vec{E}\cdot\hat{n}$ is independent of the integration variables:

Note that $\vec{E}\cdot\hat{n}$ is a constant

$$\Rightarrow \int \vec{E}\cdot d\vec{A} = \int (\vec{E}\cdot\hat{n})\, dA$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{Q}{R^2}\underbrace{\int dA}$$

would be a combination of $d\phi\, d\theta$
but we know surface area of sphere

$$= \frac{1}{4\pi\varepsilon_0}\frac{Q}{R^2}(4\pi R^2)$$

$$\boxed{\phi_E = \frac{Q}{\varepsilon_0}}$$

⊛ This is quite interesting because the flux is <u>independent of R!</u>

⊛ It turns out that the electric flux through <u>any</u> <u>closed surface</u> containing this charge will be

$$\phi_E = \frac{Q}{\varepsilon_0} \quad (\text{Essence of } \underline{\text{Gauss's Law}})$$

4.3) Gauss's Law

⊗ Starting from Coulomb's Law

$$F_{12} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}$$

it is possible to rigorously prove the following:

$$\phi_E = \oint_S \vec{E}\cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0}$$

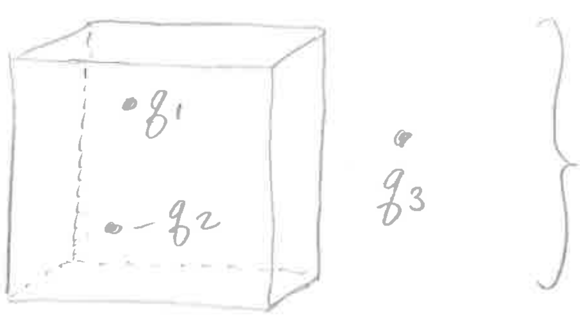
"enclosed charge"

Key Point 1: The surface S must be a closed surface

(that is what the $\oint$ stands for)

closed   closed   open   open

Key Point 2: q_{enc} is only the charge that is actually inside the closed surface.



$$q_{enc} = q_1 - q_2$$

Question: How does the flux through the two surfaces below compare?

(a) $\oint_{S_1} \vec{E}\cdot d\vec{A} > \oint_{S_2} \vec{E}\cdot d\vec{A}$ (b) $\oint_{S_1} \vec{E}\cdot d\vec{A} < \oint_{S_2} \vec{E}\cdot d\vec{A}$

(c) $\oint_{S_1} \vec{E}\cdot d\vec{A} = \oint_{S_2} \vec{E}\cdot d\vec{A}$

Answer: (c) The two surfaces enclose the same net charge $q_{enc}^{(1)} = q = q_{enc}^{(2)}$ and therefore the flux through each is the same.

* Actually calculating the electric flux for complicated charge distributions and complicated surfaces can be enormously difficult!

* Nevertheless, Gauss's Law says that we just need to know the charge enclosed by the surface

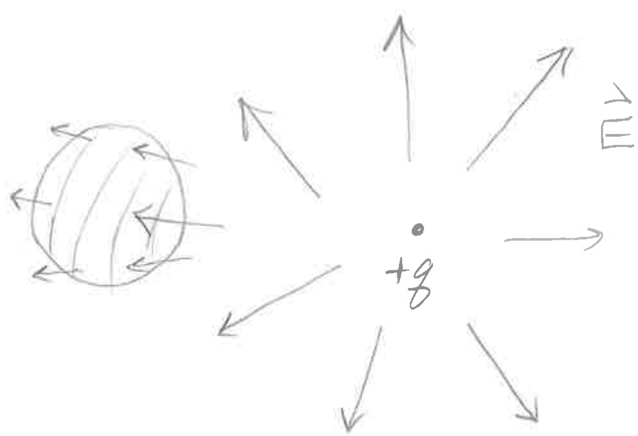
4.4) Examples of Gauss's Law

Question: What is the total electric flux through the surface below?

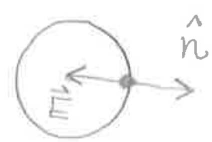
Answer: From Gauss's Law,

apparently $\phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0} = 0$.

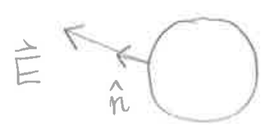
How can we visualize this?



Electric field goes in $\left(\int \vec{E} \cdot d\vec{A} < 0 \right)$



Electric field also goes out $\left(\int \vec{E} \cdot d\vec{A} > 0 \right)$



Equal flux in as flux out $\Rightarrow \oint \vec{E} \cdot d\vec{A} = 0$