PHYS 206 Lecture 3

3.1) Position, velocity, acceleration in two dimensions

3.2) Projectile motion

3.1) $\vec{r}, \vec{v}, \vec{a}$ in 2D

⊛ In the last chapter we showed that it is very convenient to decompose vectors into their x and y components

→ We will do the same for the three basic vectors $\{\vec{r}, \vec{v}, \vec{a}\}$ used to describe motion

$$\vec{r} = x\hat{i}_x + y\hat{i}_y$$

$$\vec{v} = \frac{d}{dt}\underbrace{\left[x\hat{i}_x + y\hat{i}_y\right]}$$

Question: Which of these 4 quantities can depend on time?

Answer: Only x and y (not $\hat{i}_x$ and $\hat{i}_y$)

$$\boxed{\vec{v} = \frac{dx}{dt}\hat{i}_x + \frac{dy}{dt}\hat{i}_y}$$

⊛ Note that the x and y components do not mix!

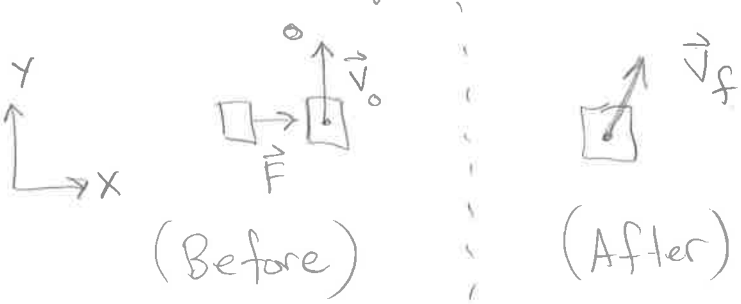
→ Same for acceleration:

$$\boxed{\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2x}{dt^2}\,\hat{\imath}_x + \frac{d^2y}{dt^2}\,\hat{\imath}_y}$$

⊛ "Las Vegas Principle": What happens in the x direction stays in the x direction (and likewise for y direction)

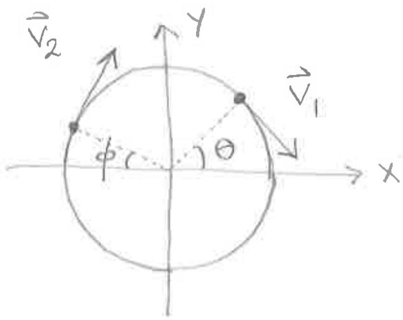
This is not <u>always</u> true (think throwing a curveball, i.e., the Magnus Force), but for most motion we consider in this course <u>we can completely decouple x motion and y motion</u>

<u>Example</u>: Suppose you are skating toward a hockey puck, and I give you a forceful check in the perpendicular direction. You will continue moving forward with the same initial component but also a new perpendicular component.



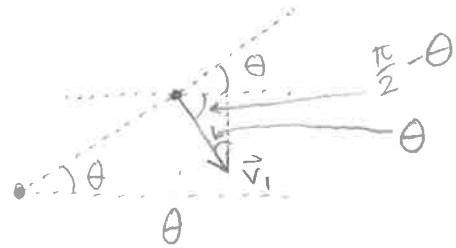
$$\vec{v}_f = v_{new}\,\hat{\imath}_x + \underbrace{v_0\,\hat{\imath}_y}_{\text{y-component unchanged}}$$

Example: Suppose a car travels at constant speed v_0 around a circular track. At the two points shown below, what is the velocity vector of the car?



(overhead view)

Solution: From geometry we see that the components of $\vec{v}_1$ can be obtained from

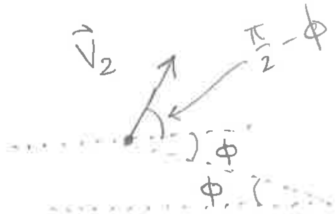


$$v_{1x} = v_0 \cos\left(\tfrac{\pi}{2}-\theta\right) = v_0 \sin\theta$$

$$v_{1y} = -v_0 \sin\left(\tfrac{\pi}{2}-\theta\right) = -v_0 \cos\theta$$

$$\underline{\vec{v}_1 = v_0 \sin\theta\, \hat{\imath}_x - v_0 \cos\theta\, \hat{\imath}_y}$$

Similarly,



$$v_{2x} = v_0 \cos\left(\tfrac{\pi}{2}-\phi\right) = v_0 \sin\phi$$

$$v_{2y} = v_0 \sin\left(\tfrac{\pi}{2}-\phi\right) = v_0 \cos\phi$$

$$\underline{\vec{v}_2 = v_0 \sin\phi\, \hat{\imath}_x + v_0 \cos\phi\, \hat{\imath}_y}$$

⊛ For constant acceleration in either the x or y direction, the same kinematic equations as before apply:

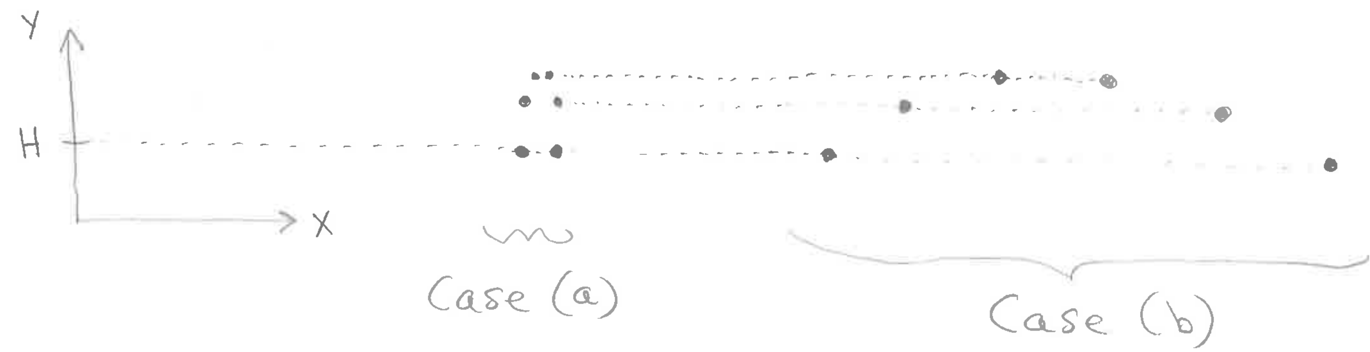
x direction	y direction
$v_x(t) = a_x t + v_{xo}$	$v_y(t) = a_y t + v_{yo}$
$x(t) = v_{xo} t + \frac{1}{2} a_x t^2 + x_o$	$y(t) = v_{yo} t + \frac{1}{2} a_y t^2 + y_o$
$v_x^2(t) = v_{xo}^2 + 2a_x(x(t) - x_o)$	$v_y^2 = v_{yo}^2 + 2a_y(y(t) - y_o)$

3.2) Projectile motion

⊛ Basic idea: launch some object on Earth and analyze subsequent motion by separating into independent x and y motions

Example: Throw ball straight up with velocity $\vec{v}_a = v_o \hat{i}_y$ vs. throw same ball across room with velocity $\vec{v}_b = v_1 \hat{i}_x + v_o \hat{i}_y$ releasing from same height H.

Since the two balls are thrown from the same height and have the same initial y velocity, their "y motion" will be exactly the same. See snapshots at six times:



At the same 6 times, the 6 heights are exactly the same and given by

$$y_a(t) = V_0 t + \frac{1}{2} a t^2 + H = y_b(t)$$

$$y_a(t) = V_0 t - \frac{1}{2} g t^2 + H = y_b(t)$$

However, in case (b) the ball also moves equal distances along x direction for equal time intervals. This is because there is no acceleration along x direction:

$$x(t) = V_1 t + x_0$$

For case (b) above, what is the trajectory of the ball in the x-y plane?

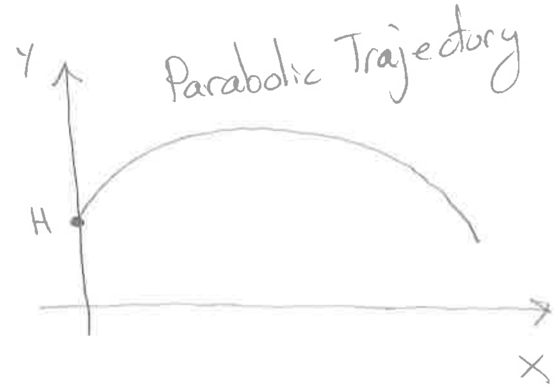
Solution: we know that $x(t) = v_1 t + x_0$. Let's define $x_0 = 0$.

Then $t = \frac{x}{v_1}$. Substitute this into expression for $y(t)$:

$$y_b(t) = v_0 t - \frac{1}{2} g t^2 + H$$

$$= v_0\left(\frac{x}{v_1}\right) - \frac{1}{2} g \left(\frac{x}{v_1}\right)^2 + H$$

$$= \left(-\frac{1}{2}\frac{g}{v_1^2}\right)x^2 + \left(\frac{v_0}{v_1}\right)x + H$$



(*) Thrown objects take parabolic paths in space.

(*) We can ask many different follow-up questions (e.g., how high does the ball travel, how far does it travel ("range"), at what time does it hit the ground, what is its speed when it hits the ground, etc.)

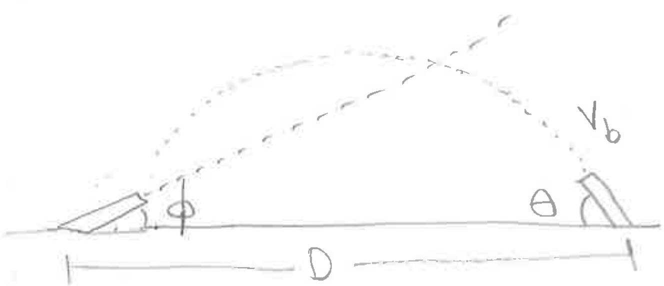
→ All of these can be answered by separately analyzing the motion in the x direction and y direction using the kinematic equations above.

(*) Note that the acceleration in either the x direction, y direction, or both can be modified. Still analyze the x and y motions separately.

Example: (Exam 1, 2011, Problem 3)

3. (33 points) A terrorist fires a cannon at you from a distance D away. The bullet has an initial velocity of magnitude v_b and the cannon is pointed at the angle θ as shown. You hope to intercept the bullet by firing a rocket at it the instant the cannon goes off. Your rocket starts at rest and is to be aimed at the angle ϕ. It is so powerful that it goes in a straight line, its acceleration having a magnitude that increases with time according to $c_1 t$, always directed at the original angle ϕ. (In other words neglect gravity for the rocket.) Obtain the equations that could be solved on a computer that determine the relationship between all the variables in order to hit the bullet.

Solution:



Compute separately x and y components of each projectile.

Bullet:

x motion	y motion
$a_x = 0$	$a_y = -g$
$V_x = -V_b \cos\theta$	$V_{y0} = V_b \sin\theta$
$x_0 = D$	$y_0 = 0$

$$\Rightarrow \quad X_b(t) = V_x t + X_0 \qquad Y_b(t) = V_{y0} t + \tfrac{1}{2} a_y t^2 + y_0$$

$$\underline{X_b(t) = -V_b \cos\theta\, t + D} \qquad \underline{Y_b(t) = V_b \sin\theta\, t - \tfrac{1}{2} g t^2}$$

Next, for the rocket

$\underline{x \text{ motion}}$

$a_x = c_1 t \cos\phi$

$V_{x0} = 0$

$X_0 = 0$

$\underline{y \text{ motion}}$

$a_y = c_1 t \sin\phi$

$V_{y0} = 0$

$Y_0 = 0$

$$\Rightarrow \quad V_x(t) = \int a_x(t)\,dt \qquad V_y(t) = \int a_y(t)\,dt$$

$$V_x(t) = \int c_1 t \cos\phi \, dt \qquad V_y(t) = \int c_1 t \sin\phi \, dt$$

$$V_x(t) = \tfrac{1}{2} c_1 t^2 \cos\phi + \cancelto{0}{c_1} \qquad V_y(t) = \tfrac{1}{2} c_1 t^2 \sin\phi + \cancelto{0}{c_2}$$

$$\Rightarrow \quad X_r(t) = \int V_x(t)\,dt \qquad Y_r(t) = \int V_y(t)\,dt$$

$$X_r(t) = \int \tfrac{1}{2} c_1 t^2 \cos\phi \, dt \qquad Y_r(t) = \int \tfrac{1}{2} c_1 t^2 \sin\phi \, dt$$

$$X_r(t) = \tfrac{1}{6} c_1 t^3 \cos\phi + \cancelto{0}{c_3} \qquad Y_r(t) = \tfrac{1}{6} c_1 t^3 \sin\phi + \cancelto{0}{c_4}$$

In order for the rocket to hit the bullet, they must have the same x and y coordinates at the same time:

$$x_b(t) = x_r(t) \qquad\qquad y_b(t) = y_r(t)$$

$$-v_b t\cos\theta + D = \tfrac{1}{6}c_1 t^3 \cos\phi \qquad\qquad v_b t \sin\theta - \tfrac{1}{2}g t^2 = \tfrac{1}{6}c_1 t^3 \sin\phi$$

(Two unknowns: t, ϕ)

You should be able to solve the following problems from previous Exam 1's:

2005 (2)	2009 (2)	2013 (3)
2005 (3)	2011 (3)	2013 (4)
2006 (3)	2012 (1c)	2014 (1)
2007 (3)	2012 (2)	2014 (3)
2008 (4)	2013 (1)	2014 (4)