3. Electric Potential

3.1) Electric potential energy

(*) Any fixed set of charges gives rise to a conservative electrostatic force

This follows from Coulomb's Law for the force between a charge q_1 at the origin and a charge q_2 at $\vec{r}$:

$$\vec{F}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2} \hat{\iota}_r$$

(Exact same form as gravity: $\vec{F}(\vec{r}) = -G\frac{m_1 m_2}{r^2} \hat{\iota}_r$,

which we showed had potential energy function

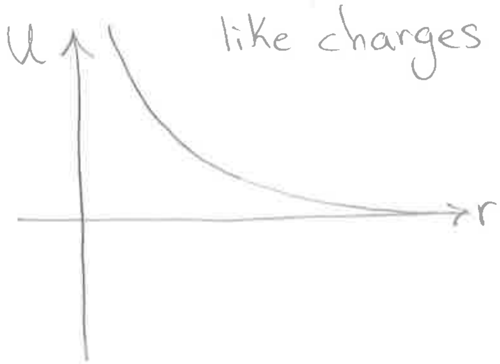
$\mathcal{U}(r) = -G\frac{m_1 m_2}{r}$)

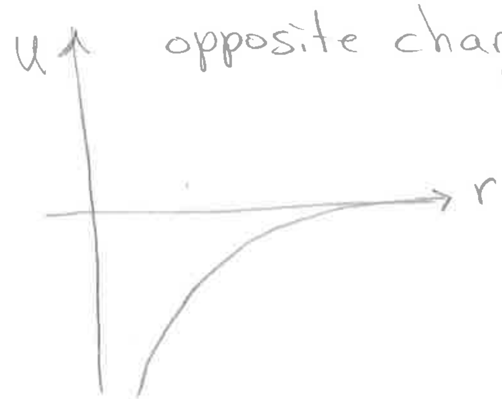
Repeat exact same calculation to get

$$\boxed{U_e(r) = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r}}$$

(only difference is sign → gravity always attractive, but electric force can also be repulsive)

(*) To be crystal clear, $U_e(\vec{r})$ above is the electric potential energy between two charges q_1 and q_2 located a distance r apart and choosing the zero of potential energy at ∞ : $U(r=\infty) \equiv 0$.





Bring same-charge particles together
⇒ increase U

Bring opposite-charge particles together
⇒ decrease U

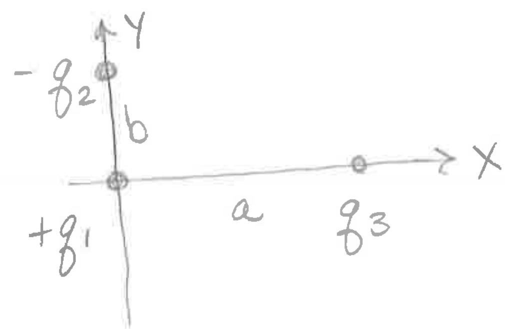
(*) Key Point: Potential energies from different sources just add up.

Recall that in mechanics we could drop an object that is attached to a spring and use energy conservation to find speed:

$$KE(y_1) + \underbrace{U_s(y_1) + U_g(y_1)}_{\text{Just added spring and gravitational potential energies to get total } U.} = KE(y_2) + U_s(y_2) + U_g(y_2)$$

(*) For multiple point charges, just add up separate electric potential energies

<u>Example</u>: What is the potential energy of charge q_3 in the vicinity of charges q_1 and q_2 below?



<u>Solution</u>: Just add U_{13} and U_{23} to get

$$U_{13} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_3}{a}$$

$$U_{23} = \frac{1}{4\pi\varepsilon_0}\frac{(-q_2)(q_3)}{\sqrt{a^2+b^2}}$$

$$U_{total} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_3}{a} - \frac{q_2 q_3}{\sqrt{a^2+b^2}}\right)$$

So much easier than dealing with electric forces! Am I right?

3.2) Electric potential

Wait, wasn't this the title of the previous section??

No!

$$\boxed{\text{Electric potential} \neq \text{Electric potential energy}}$$

Recall that electric field allowed us to describe how a given charge affects the space surrounding it $\left(\vec{E} \equiv \frac{\vec{F}_q}{q}\right)$

Likewise we define the electric potential in the area around a point charge to be

$$V_1(r) = \frac{U_{12}(r)}{q_2} = \frac{1}{q_2}\left[\frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r}\right]$$

$$\Rightarrow \boxed{V(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}}$$

⊛ We say that the charge q creates an electric potential that depends only on the charge q and the distance r away from q.

* Positive charges create positive potentials and negative charges create negative potentials.

* Given the electric potential $V(r)$, we can easily calculate the electric potential energy of a point charge:

$$\boxed{U_q(r) = qV(r)}$$

* There are 2 ways to compute the electric potential

1. From the fundamental definition:

$$\underbrace{V(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}}_{\text{Point charges: add up}} \rightarrow \underbrace{dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r}}_{\text{Charge distributions: integrate}}$$

2. From the electric field $\vec{E}(\vec{r})$:

$$V(b) - V(a) = -\int_a^b \vec{E}\cdot d\vec{r}$$

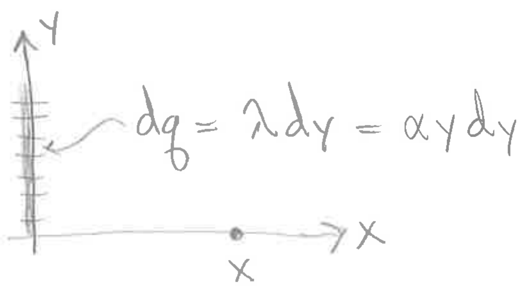
(Exactly analogous to $U(b) - U(a) = -\int_a^b \vec{F}\cdot d\vec{r}$)

* Use 1 if given the distribution of charges and 2 if you are given the electric field $\vec{E}$.

Example: Suppose there is rod of length L placed on the Y-axis with one end at the origin. The rod has a non-uniform charge density $\lambda(y) = \alpha y$, where α is a positive constant. Find the electric potential along the X-axis.

Solution: Note... we could first calculate $\vec{E}$ and then integrate to get V, but that is overkill. Instead, just use fundamental definition $dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r}$.

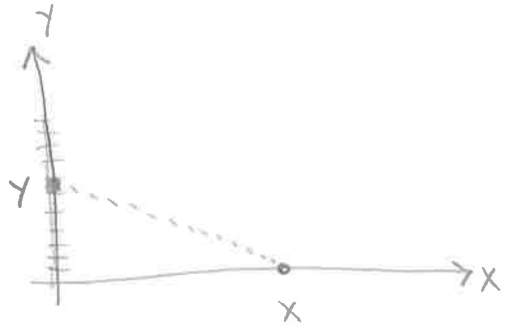
(*) Follow same steps as with computing $\vec{E}$ field, except we can drop the step where we broke $\vec{E}$ into components.

(1) 

Break up into pieces and determine dq.

(2) Integration region: $\int_0^L dy$

(3) Take arbitrary point in integration region and compute dV:



$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r}$$

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dy}{\sqrt{x^2+y^2}}$$

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{\alpha y\, dy}{\sqrt{x^2+y^2}}$$

(4) Just integrate:

$$V(x) = \int dV = \int_0^L \frac{1}{4\pi\varepsilon_0}\frac{\alpha y}{\sqrt{x^2+y^2}}\, dy$$

$$= \frac{\alpha}{4\pi\varepsilon_0}\int_0^L \frac{y}{\sqrt{x^2+y^2}}\, dy$$

Again, we will give you any non-trivial integrals.

Hopefully you recognize that this is much easier than computing the electric field!

<u>Example</u>: Suppose the electric field in a region of space is given piecewise by

$$\vec{E}(r) = \begin{cases} K\frac{r}{R^3}\hat{\imath}_r & \text{for } r<R \\ K\frac{1}{r^2}\hat{\imath}_r & \text{for } r>R \end{cases}$$

Find the potential difference $V(2R)-V(0)$.

Solution: Since we are given the electric field,

we use $V(2R) - V(0) = -\int_0^{2R} \vec{E} \cdot d\vec{r}$

$$= -\int_0^R \vec{E}_1 \cdot d\vec{r} - \int_R^{2R} \vec{E}_2 \cdot d\vec{r}$$

Since $\vec{E}$ has different functional form, we break into 2 regions.

$$= -\int_0^R K \frac{r}{R^3}\, dr - \int_R^{2R} K \frac{1}{r^2}\, dr$$

$$= -\frac{K}{R^3}\left[\frac{1}{2}r^2\right]_0^R - K\left[-\frac{1}{r}\right]_R^{2R}$$

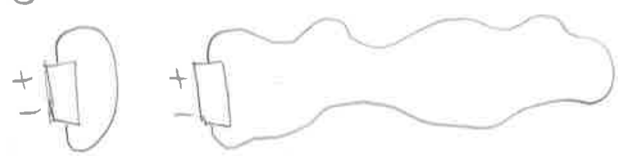
$$= -\frac{K}{R^3}\frac{1}{2}R^2 + K\left[\frac{1}{2R} - \frac{1}{R}\right]$$

$$= -\frac{1}{2}\frac{K}{R} + K\left[\frac{-1}{2R}\right] = \boxed{-\frac{K}{R}}$$

Sanity check: Since we are moving in the direction of $\vec{E}$ field in going from 0 to 2R, the potential should be decreasing and $V(2R) - V(0) < 0$, which is what we found.

⊛ Like changes in potential energy, changes in electric potential are path independent.

Very important for circuits!



Bend wires however we want!

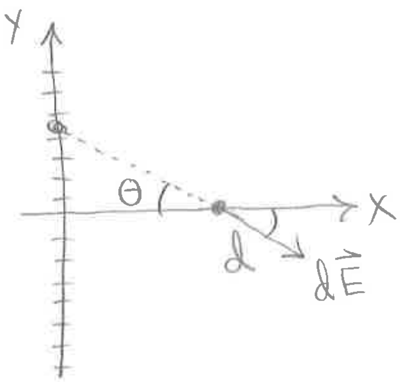
3.3) Electric potential for infinite line charge

(*) When integrating $dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r}$ we have assumed that

$V(\infty) \equiv 0.$

Sometimes we cannot make this assumption

Example: Suppose we have an infinitely long line charge with constant charge density λ_0. What is the electric field a distance d away?



① $dq = \lambda_0\, dy$

② $\int_{-\infty}^{\infty} dy$

③ $dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda_0\, dy}{y^2+d^2}$

④ $E_y = 0$ by symmetry

$$dE_x = \frac{1}{4\pi\varepsilon_0}\frac{\lambda_0\, dy}{y^2+d^2}\cos\theta = \frac{1}{4\pi\varepsilon_0}\frac{\lambda_0\, dy}{y^2+d^2}\frac{d}{\sqrt{y^2+d^2}}$$

⑤ $E_x = \frac{1}{4\pi\varepsilon_0}\lambda_0 d \underbrace{\int_{-\infty}^{\infty}\frac{dy}{(y^2+d^2)^{3/2}}}_{2/d^2}$ ($y = \tan u$ substitution)

$\boxed{E_x = \frac{\lambda_0}{2\pi\varepsilon_0 d}}$ Perfectly well behaved.

What about electric potential at $x = d$, assuming we choose $V(\infty) \equiv 0$?

$$V(x) - V(\infty) = V(x) = -\int_{\infty}^{x} \vec{E} \cdot d\vec{r}$$

$$= -\int_{\infty}^{x} \left(\frac{\lambda_0}{2\pi \varepsilon_0 x} \hat{\imath}_x \right) \cdot \left(dx\, \hat{\imath}_x + dy\, \hat{\imath}_y \right)$$

$$= -\int_{\infty}^{x} \frac{\lambda_0}{2\pi \varepsilon_0} \frac{1}{x}\, dx$$

Problem!

$$= -\frac{\lambda_0}{2\pi \varepsilon_0} \left[\ln x \right]_{\infty}^{x} = -\frac{\lambda_0}{2\pi \varepsilon_0} \left[\ln x - \ln \infty \right]$$

Electric potential at any point along x-axis would be infinite!

(*) Sometimes we cannot choose $V(\infty) \equiv 0$, particularly if the charge distribution extends to ∞.

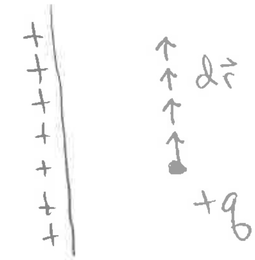
(*) However, changes in electric potential are always well defined:

$$V(b) - V(a) = -\frac{\lambda_0}{2\pi \varepsilon_0} \left[\ln b - \ln a \right] = \boxed{\frac{\lambda_0}{2\pi \varepsilon_0} \ln \frac{a}{b}}$$

3.4) Equipotential surfaces

What happens if we move a point charge upward and parallel to a very long line charge?

(a) potential energy increases
(b) potential energy decreases
(c) potential energy stays the same



Answer: (c) because $U(b) - U(a) = -\int_a^b \vec{F} \cdot d\vec{r}$ and

$\vec{F}$ is perpendicular to $d\vec{r} \Rightarrow \vec{F} \cdot d\vec{r} = 0$.

We could move point charge anywhere on a cylinder centered on the line charge without doing work.

$\Rightarrow$ Cylinder is called an equipotential surface

⊛ Conservative forces do <u>no</u> work along equipotential surfaces:

$$W = \Delta KE = -\Delta U = 0$$

<u>Question</u>: What are the equipotential surfaces for a point charge?

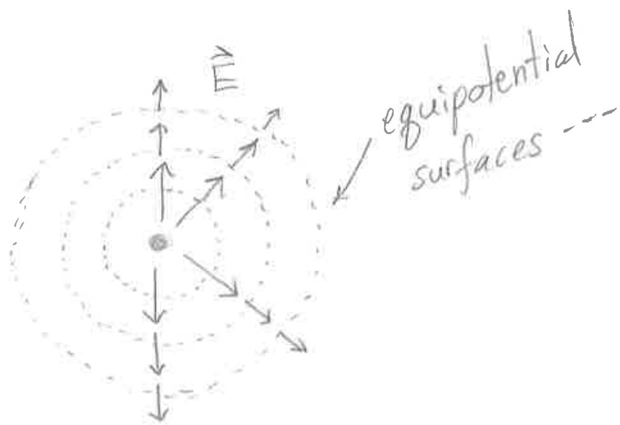
Answer: Concentric spheres centered at point charge

What is the relationship between equipotential surfaces and the electric field?

(*) Equipotential surfaces always perpendicular to $\vec{E}$ field

(by definition, no work is done $\Rightarrow \vec{F} \cdot d\vec{r} = 0$

$\Rightarrow q\vec{E} \cdot d\vec{r} = 0 \Rightarrow \underline{\vec{E} \cdot d\vec{r} = 0}$.)



(*) Plotting equipotential surfaces analogous to elevation maps (topographic maps) for terrain.

Actually, more than just an analogy: curves on topographic maps are the equipotential surfaces for the gravitational force