2. Electric Fields

2.1) What are electric fields?

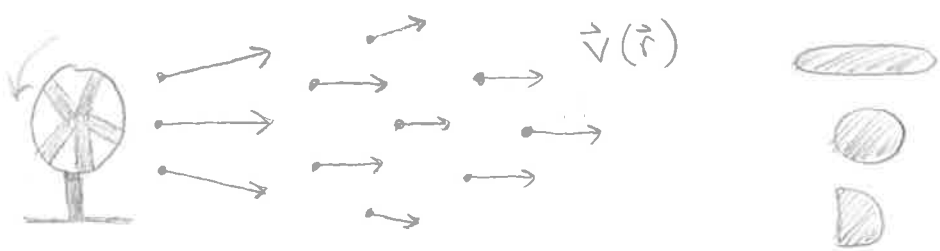
2.2) Electric fields generated by point charges

2.3) Charge distributions

2.4) Electric fields due to charge distributions

2.1) What are electric fields?

First, a simple analogy: If you turn on a powerful fan in a wind tunnel, it will create a velocity field.

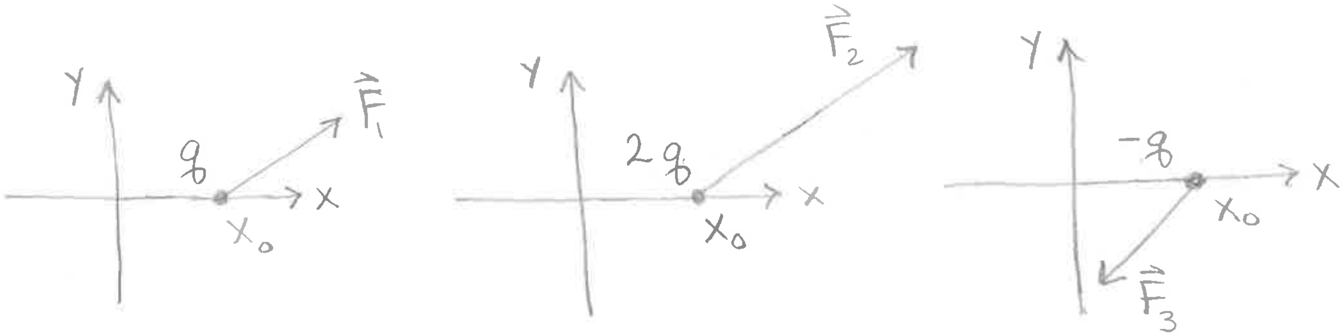


Different shaped objects experience different forces in this velocity field.

(*) The background velocity field exists whether or not we put an object in the wind tunnel where it would experience a force.

(*) Just like the fan produces a velocity field $\vec{v}(\vec{r})$ that exerts a force on other objects, we say that a system of charges produces an electric field $\vec{E}(\vec{r})$ that can exert a force on other charged objects.

Suppose we have an electric field $\vec{E}(\vec{r})$ located at the position x_0. How will particles with different charge behave?



(1) test charge (2) twice the charge (3) negative charge

Since all electric forces are proportional to the charge q that we place at x_0, we can define

$$\boxed{\vec{E}(x_0) \equiv \frac{\vec{F}_q(x_0)}{q}}$$

force exerted on a charge q at x_0

⊛ With this definition,

$$\vec{E}_1(x_0) = \vec{E}_2(x_0) = \vec{E}_3(x_0)$$ and $\boxed{\vec{E} \text{ is uniquely defined}}$.

Question: In the above example, in which direction would the electric field at x_0 point (assume $q > 0$)?

(a) $\longrightarrow$ (b) $\longleftarrow$ (c) $\nearrow$ (d) $\swarrow$

Answer: (c) Since $q > 0$, the direction of $\vec{E}(x_0)$ will be the same as forces $\vec{F}_1$ and $\vec{F}_2$ but opposite to the direction of $\vec{F}_3$.

2.2) Electric fields generated by point charges

⊛ From Coulomb's Law, $F = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}$, the electric field due to charge q_1 is given by

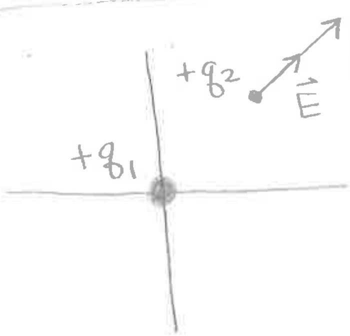
$$E_1(r) = \frac{1}{q_2} F_{12} = \frac{1}{\cancel{q_2}}\left(\frac{1}{4\pi\varepsilon_0}\frac{q_1 \cancel{q_2}}{r^2}\right)$$

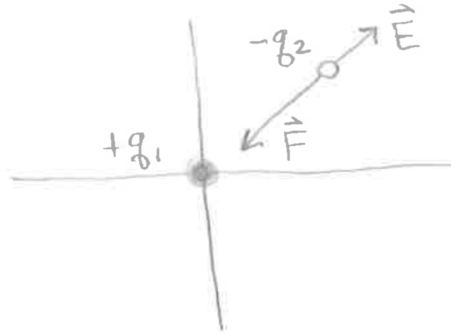
$$\boxed{E_{q_1}(r) = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r^2}}$$

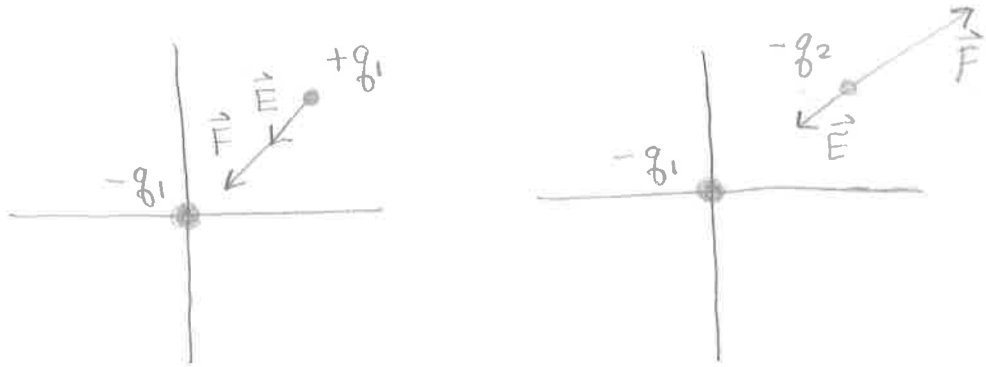
"r" is the distance from q_1

⊛ Direction of electric field: from the definition $\vec{E}_1 = \frac{\vec{F}_{12}}{q_2}$,

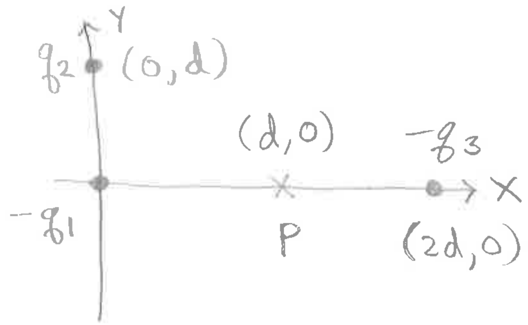
> electric fields point away from ⊕ charges and toward ⊖ charges





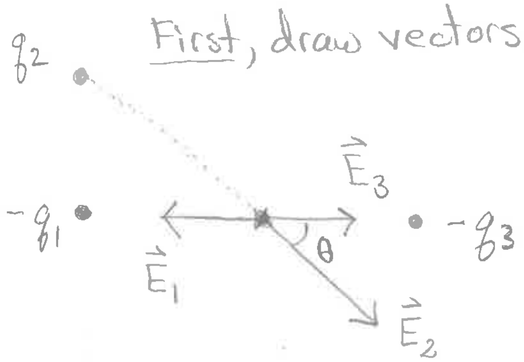


Question: Consider the system of three point charges shown below. What is the electric field produced at the point P shown in the figure?



Solution: Just like forces add as vectors, we can compute $\vec{E}_1(p)$, $\vec{E}_2(p)$, and $\vec{E}_3(p)$ and then just add as vectors.

First, draw vectors.



$$\left.\begin{aligned} E_1(p) &= \frac{1}{4\pi\varepsilon_0}\frac{q_1}{d^2} \\ E_2(p) &= \frac{1}{4\pi\varepsilon_0}\frac{q_2}{(\sqrt{2}d)^2} \\ E_3(p) &= \frac{1}{4\pi\varepsilon_0}\frac{q_3}{d^2} \end{aligned}\right\} \text{Second, compute magnitudes of electric fields.}$$

Third, break up into components:

$$E_{1x} = (-)\frac{1}{4\pi\varepsilon_0}\frac{q_1}{d^2}$$

$$E_{3x} = (+)\frac{1}{4\pi\varepsilon_0}\frac{q_3}{d^2}$$

$$E_{2x} = (+)\frac{1}{4\pi\varepsilon_0}\frac{q_2}{2d^2}\cos\theta = \frac{1}{4\pi\varepsilon_0}\frac{q_2}{2d^2}\frac{1}{\sqrt{2}}$$

$$E_{2y} = (-)\frac{1}{4\pi\varepsilon_0}\frac{q_2}{2d^2}\sin\theta = -\frac{1}{4\pi\varepsilon_0}\frac{q_2}{2d^2}\frac{1}{\sqrt{2}}$$

$$E_x^{total} = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_3}{d^2} - \frac{q_1}{d^2} + \frac{q_2}{2\sqrt{2}\,d^2}\right]$$

$$E_y^{total} = \frac{1}{4\pi\varepsilon_0}\left[\frac{-q_2}{2\sqrt{2}\,d^2}\right]$$

⊛ Important Point: We could still define the electric field even if the electric force were very different. As long as $F_{12} \sim q_1 q_2$, we can always define $E_1 = \frac{F_{12}}{q_2}$.

Examples:

$$F_{12} = K\frac{q_1 q_2}{r^4} \longrightarrow E_1 = K\frac{q_1}{r^4}$$

$$F_{12} = K q_1 q_2 \ln(r) \longrightarrow E_1 = K q_1 \ln(r)$$

$$F_{12} = K q_1 q_2\, g(r) \longrightarrow E_1 = K q_1\, g(r)$$

2.3) Charge distributions

If I rub a balloon against my hair, many charges get distributed over the surface of the balloon.

Rub just one side of balloon on hair

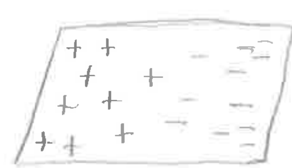
Rub entire balloon all over my hair

⊛ In these types of situations where there are too many individual charges to keep track of, we say that there is a "charge distribution" on the balloon.

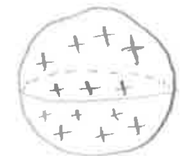
⊛ There are <u>three</u> main types of charge distributions:

(a) Linear (charges spread on a line)

(b) Surface (charges spread on 2D surface)



(c) Volume (charges spread throughout 3D volume)



For now, let's just focus on linear charge distributions.

"Uniform" charge distributions have charges equally spread over the line : + + + + + + + + + + + + + +

"Non-uniform" charge distributions have varying charge concentrations : + + + + + + + + +

⊛ We can describe both situations with a "linear charge density"

$$\lambda(x) \equiv \frac{dq}{dx} = \frac{\text{charge}}{\text{length}} \quad \left(\text{similar to density} = \frac{\text{mass}}{\text{Volume}}\right)$$

Example : $\lambda(x) = \lambda_0$ (constant, uniform charge density)

$\lambda(x) = \alpha x$ (non-uniform, increases with x)

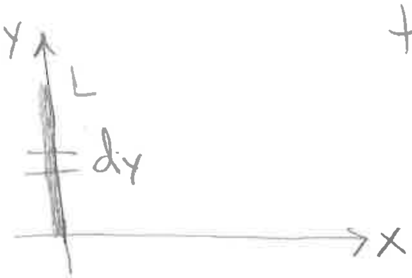
⊛ Given the linear charge density $\lambda(x)$, we compute the charge as follows :

$$\boxed{dq = \lambda(x)\,dx}$$ (small amount of charge in a small segment of line dx)

$$\boxed{Q = \int_a^b \lambda(x)\,dx}$$ (total amount of charge on line between points "a" and "b")

Example: Suppose we have a total charge Q equally spread over a thin rod of length L placed at the origin along the y-axis. How much charge lies within a segment of length dy?

Answer: Since Q is uniformly spread over length L, the linear charge density is given by

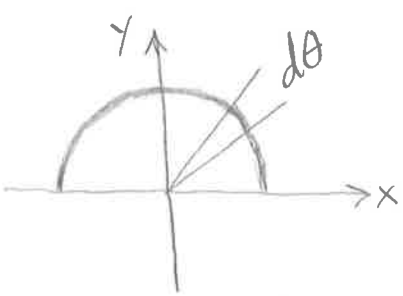


$$\lambda(y) = \frac{Q}{L} = (\text{constant}).$$

Therefore $dq = \lambda(y)\,dy = \boxed{\frac{Q}{L}\,dy}$

Example: Suppose a total charge Q is spread over a semicircular arc of radius R. How much charge is enclosed in a small arc of angle $d\theta$?

Answer: First, note that again $\lambda(\theta) = \frac{Q}{\pi R} = (\text{constant})$.



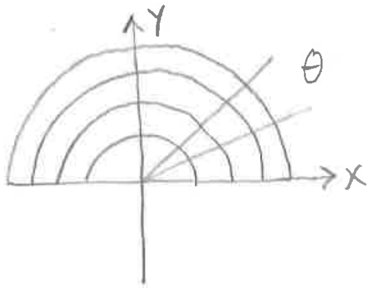
But $dq \neq \lambda(\theta)\,d\theta$! Why not ??

The units wouldn't work:

$$\frac{\text{charge}}{\text{length}} \times \text{angle} \neq \text{charge}$$

(*) We <u>must</u> multiply linear charge density by a <u>length</u> in order to get charge.

What is the arclength of the segment?



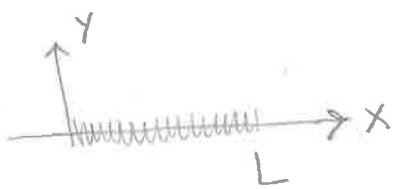
Arclength increases proportional to R for fixed angle θ: $\boxed{S = R\theta}$

Example	Total circle	Semicircle	Small angle
Angle (θ)	2π	π	$d\theta$
Arclength (s)	$2\pi R$	πR	$R\,d\theta$

$$\Rightarrow \quad dq = \lambda(\theta)\,ds = \left[\frac{Q}{\pi R}\right](R\,d\theta) = \boxed{\frac{Q}{\pi}\,d\theta}$$

<u>Question:</u> What if the charge density is <u>not</u> uniform?

<u>Example:</u>



$$\lambda(x) = \lambda_0 \frac{x}{L}$$

<u>Answer:</u> Solution is just as easy, except dq now depends on x.

$$dq = \lambda(x)\,dx \quad \text{(Always!)}$$

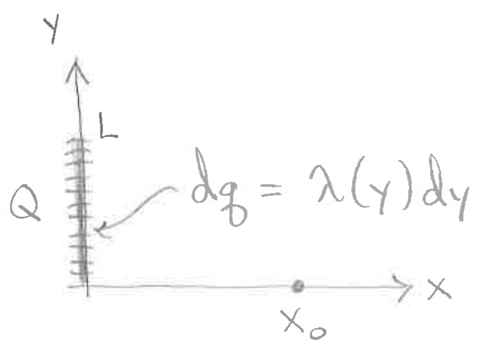
$$\Rightarrow \quad dq = \lambda(x)dx = \boxed{\lambda_0 \frac{x}{L}\,dx}$$

⊛ This is important for computing the electric field due to a linear charge distribution $\Rightarrow$ requires integration.

Question : What is the electric field at any point x_0 along the positive x-axis for vertical line charge considered previously?

Solution : Always follow these 5 steps:

① Break up the physical line charge into small pieces of length ds. The charge constained within ds is always $\boxed{dq = \lambda(s)\, ds}$

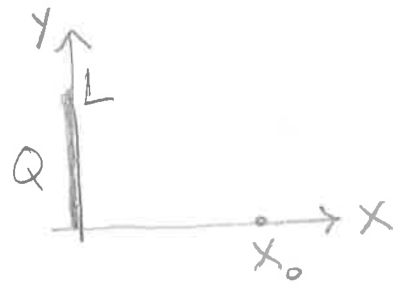


⊛ $ds \equiv \{dx, dy, R\,d\theta, \ldots\}$

for charge spread over $\{$x-axis, y-axis, circle, ...$\}$.

Each small segment of charge dq will contribute to the electric field at x_0 but in different ways as we move along the physical length of the rod.

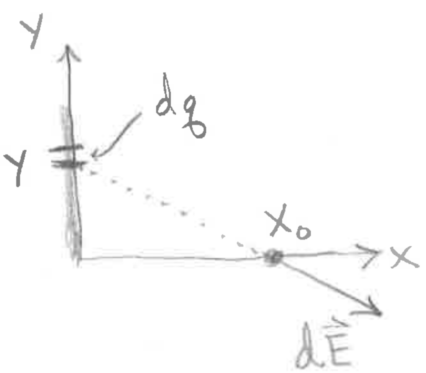
(2) Define the integration region. The integration must be over the physical length of the charge distribution:



Rod extends along y-axis from 0 to L

$$\Rightarrow \int_0^L dy \quad (\text{not integrating over } x\,!!)$$

(3) Choose an arbitrary point s within integration region and compute the magnitude of the small electric field it produces at the desired point (here x_o):



$$dE = \frac{1}{4\pi\varepsilon_o}\frac{dq}{r^2}$$

In general either dq or r can depend on integration variable y.

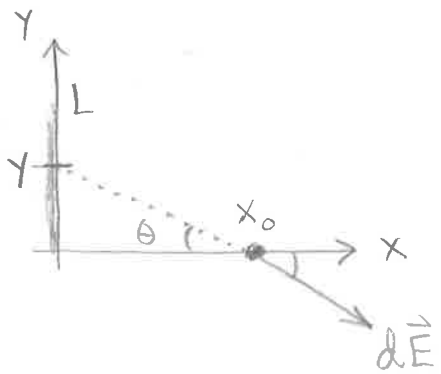
$$dE = \frac{1}{4\pi\varepsilon_o}\frac{\lambda\, dy}{x_o^2 + y^2}$$

Recall $\lambda(y) = \frac{Q}{L}$ = (constant).

$$dE = \frac{1}{4\pi\varepsilon_o}\frac{(Q/L)\, dy}{x_o^2 + y^2}$$

(4) Before we integrate, we <u>must</u> decompose dE into its <u>vector components</u> (just like we have done all along for electric forces and fields due to point charges).

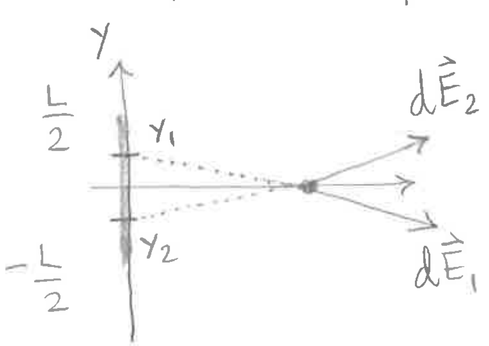
(*) Pay close attention to whether <u>symmetry requires</u> that one (or more) components of $\vec{E}$ are 0.



$$dE_x = dE\cos\theta$$
$$= \left(\frac{1}{4\pi\varepsilon_0}\frac{Q}{L}\frac{dy}{y^2+x_0^2}\right)\frac{x_0}{\sqrt{y^2+x_0^2}}$$

$$dE_y = dE\sin\theta$$
$$= \left(\frac{1}{4\pi\varepsilon_0}\frac{Q}{L}\frac{dy}{y^2+x_0^2}\right)\frac{y}{\sqrt{y^2+x_0^2}}$$

(*) Note that here symmetry doesn't help us, but if charge distribution were symmetric about the origin, then the y-component of $\vec{E}$ field would vanish:



$$d\vec{E}_{2y} = \uparrow$$
$$d\vec{E}_{1y} = \downarrow$$
(cancel)

(5) Finally, integrate components separately :

$$E_x = \int dE_x = \int_0^L \frac{1}{4\pi\varepsilon_0} \frac{Q}{L} x_0 \frac{dy}{(y^2+x_0^2)^{3/2}}$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{Q}{L} x_0 \int_0^L \frac{dy}{(y^2+x_0^2)^{3/2}}$$

We will give you any nontrivial integrals.

$$E_y = \int dE_y = \int_0^L \frac{1}{4\pi\varepsilon_0} \frac{Q}{L} \frac{y}{(y^2+x_0^2)^{3/2}} dy$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{Q}{L} \int_0^L \frac{y\,dy}{(y^2+x_0^2)^{3/2}}$$