# 2. Mechanics Review

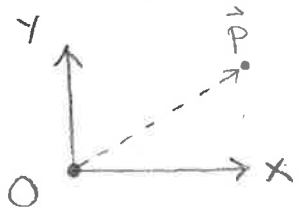
2.1 Polar coordinates

2.2 Potential energy functions

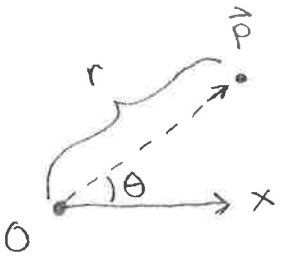
## 2.1) Polar coordinates

Most of you have dealt with functions $f(r,\theta)$ in polar coordinates. But more tricky are vector fields $\vec{v}(r,\theta)$ in polar coordinates.

To define Cartesian or polar coordinates, first set the location of the origin and then set the direction of the x-axis:



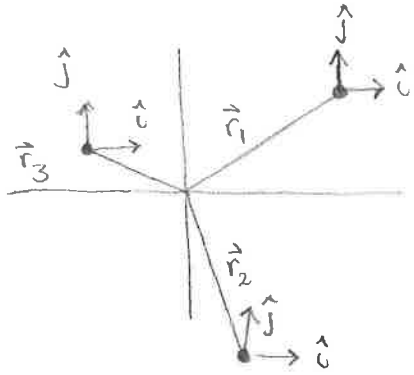

Cartesian coordinates: y-direction automatically fixed to be perpendicular to x-axis




Polar coordinates: angle $\theta$ is defined relative to x-axis and r is the magnitude of the vector

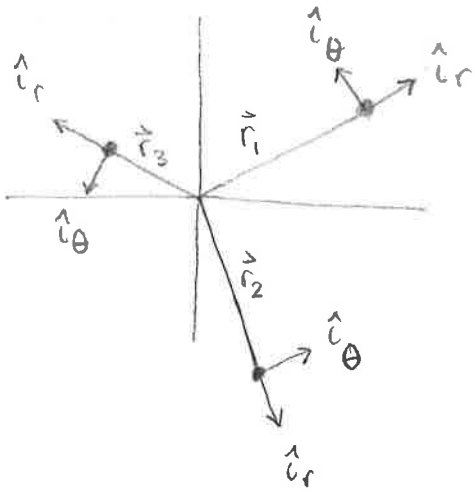
Polar coordinates are often more convenient than Cartesian coordinates, but they have a tricky feature for vector fields:

$$\boxed{\text{Basis vectors } \hat{i}_r \text{ and } \hat{i}_\theta \text{ depend on location } \vec{r}}$$




Cartesian coordinates:

Both $\hat{i}$ and $\hat{j}$ are fixed regardless of the location $\{\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots\}$ at which we want to place a vector (say, $\vec{F}(\vec{r})$)




Polar coordinates:

Both $\hat{i}_r$ and $\hat{i}_\theta$ change orientation depending on the location $\{\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots\}$ at which we want to place a vector

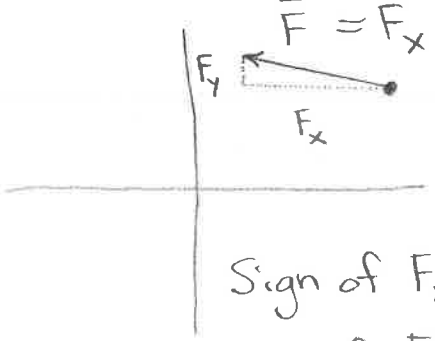
In PHYS 218 we showed that the two bases are related by

$$\hat{i}_r = \cos\theta\, \hat{i} + \sin\theta\, \hat{j}$$

$$\hat{i}_\theta = -\sin\theta\, \hat{i} + \cos\theta\, \hat{j}$$

Suppose we apply a force $\vec{F}$ at location $\vec{r}_1$ above

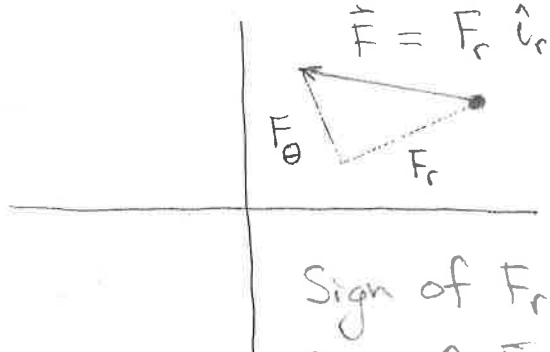
$$\vec{F} = F_x\, \hat{i} + F_y\, \hat{j}$$




Sign of $F_x$?

Sign of $F_y$?

$$\vec{F} = F_r\, \hat{i}_r + F_\theta\, \hat{i}_\theta$$




Sign of $F_r$?

Sign of $F_\theta$?

⊛ If we place the origin at the location of a point charge $q_1$, then the force on any other nearby charge $q_2$ due to $q_1$ is given by

$$\vec{F}_2 = \underbrace{\frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}}_{\text{magnitude depends only on } r \text{ and not on } \theta} \hat{\imath}_r = F(r)\hat{\imath}_r$$

In PHYS 218 we showed that <u>any</u> force of the form

$\vec{F}(\vec{r}) = F(r)\hat{\imath}_r$ has an associated <u>potential energy function</u>.

## 2.2) Potential energy functions

⊛ Recall that potential energy makes life much simpler because total mechanical energy will be conserved:

$$KE^i + U_1^i + U_2^i + \dots = KE^f + U_1^f + U_2^f + \dots$$

⊛ If $\vec{F}$ is a conservative force, then the work $W = \int_a^b \vec{F}\cdot d\vec{l}$ done by $\vec{F}$ is <u>path independent</u>: $-W = -\int_a^b \vec{F}\cdot d\vec{l} = U(b) - U(a)$.

Then from work-Kinetic energy theorem $W = \int \vec{F}_{tot}\cdot d\vec{l} = \Delta KE$,

we find $-(U(b) - U(a)) = KE(b) - KE(a)$

$\Rightarrow \quad KE(a) + U(a) = KE(b) + U(b)$ <u>E conservation</u>

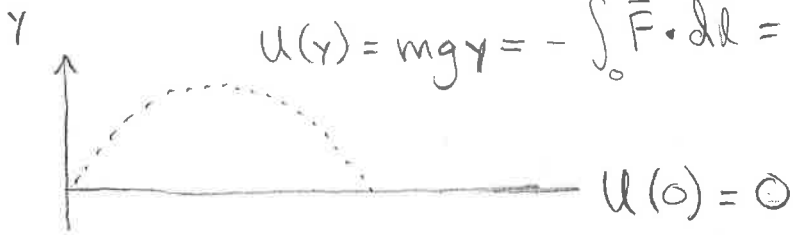
⊛ Notice that we have only derived the <u>change</u> in potential energy

⊛ To define the potential energy function we must set the location where $U=0$ (choose any convenient place)

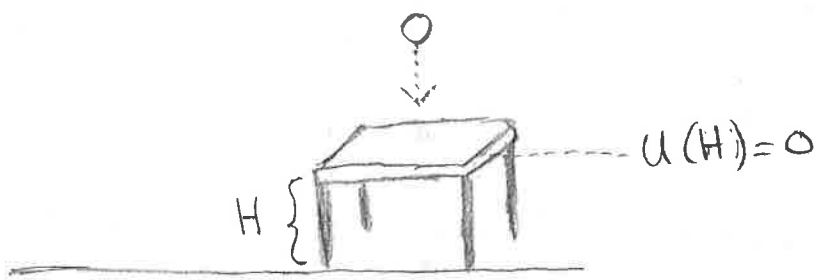
<u>Example</u>: Gravitational force

$$U(y) = mgy = -\int_0^y \vec{F}\cdot d\vec{l} = -\int_0^y (-mg)\,dy$$

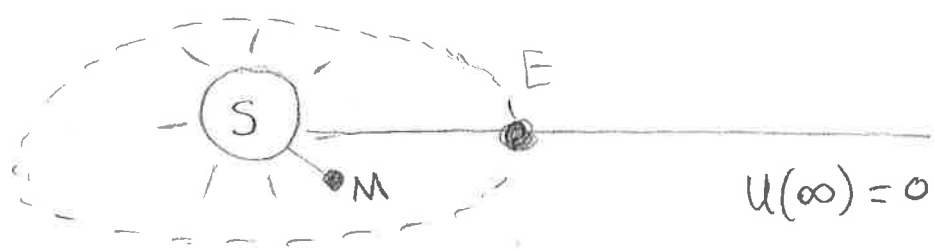
(a) Projectile motion




(b) Drop ball onto table




(c) Solar system




<u>Question</u>: Why not choose $U(0)=0$ at the center of the Sun?

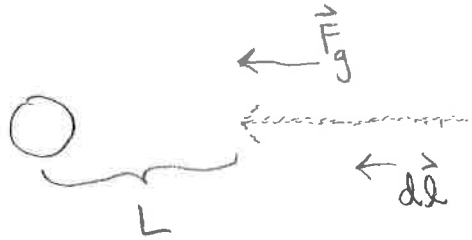
That would be a problem since $\int_0^a \underbrace{G\frac{m_1 m_2}{r^2}}_{F_g(r)} dr$ diverges!

The gravitational force is 0 infinitely far away, so it is convenient to set $U(\infty)=0$.

In this case what is the gravitational potential energy a distance L from the Sun?

$$U(L) = \underset{\text{end}}{U(L)} - \underset{\text{start}}{U(\infty)} = -\int_\infty^L \vec{F}_g \cdot d\vec{l} = -\int_\infty^L F_g\, dl = -\int_\infty^L F_g(-dr)$$

$$= \int_\infty^L G\frac{m_1 m_2}{r^2}\, dr$$




$$= \left[-\frac{Gm_1m_2}{r}\right]_\infty^L = \boxed{-\frac{Gm_1m_2}{L}}$$

(*) Warning: It is very easy to obtain the incorrect sign.

Trick 1: If your integral $-\int_a^b \vec{F}\cdot d\vec{l}$ has $b<a$, always reverse the path and you will be ok:

$$-\int_\infty^L \vec{F}_g \cdot d\vec{l} = \int_L^\infty \vec{F}_g \cdot d\vec{l} = \int_L^\infty -\frac{Gm_1m_2}{r^2}\, dr$$

Trick 2 : Sanity check

(a) If path is in direction of force, potential energy decreases

(b) If path is opposite the force, potential energy increases

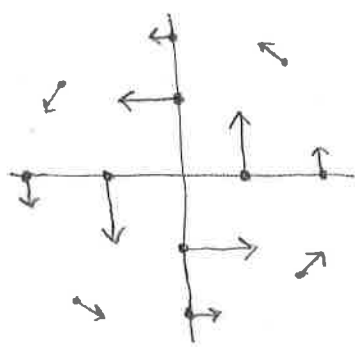
We have reviewed how to obtain $U$ from $\vec{F}$.

Much easier to obtain $\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z$ from $U(x,y,z)$:

$$F_x = -\frac{\partial U}{\partial x} \qquad F_y = -\frac{\partial U}{\partial y} \qquad F_z = -\frac{\partial U}{\partial z}$$

For a radial potential energy $U(r)$:

$$F_r = -\frac{dU}{dr}$$

What is an example of a non-conservative force?

$$\vec{F}(\vec{r}) = G\frac{m_1 m_2}{r^2}\hat{i}_\theta$$