PHYS 207 Electricity and Magnetism

14. Special Theory of Relativity

* Buried in Maxwell's equations is an even more profound realization about Nature.

* Einstein's 1905 paper on special relativity was titled "On the Electrodynamics of Moving Bodies".

Relativity is based on two key insights:

1. From Maxwell's equations, light travels at only one possible speed: $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8$ m/s.

2. Laws of physics are independent of which constant velocity reference frame you are in.

* Both of these rely on the fact that empty space is truly empty

(In the early 1900's physicists looked for evidence that there is some medium that fills space, but they found nothing.)

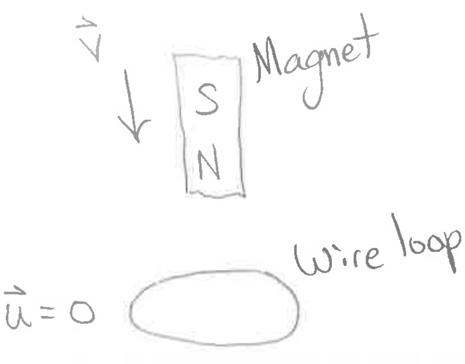
Question: If you are the only object in space, can you tell if you are moving?

Answer: No, the question doesn't even make sense really.

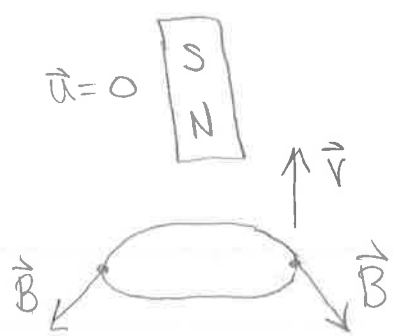
Question: If you and a friend are the only objects in space and you are moving toward each other, can you tell who is moving?

Answer: No, the situation would look the same whether you are moving toward your friend at rest or vice versa.

(*) In the case of electrodynamics, this leads to some counterintuitive results (but nothing really crazy):



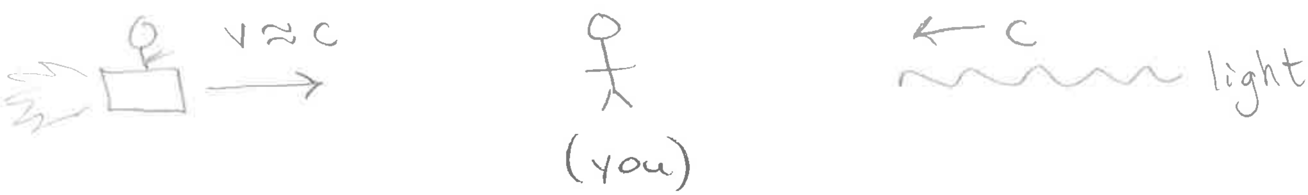
Changing $\vec{B}$ field induces an $\vec{E}$ field that drives current in loop CCW



Same relative motion, but now current flows CCW due to magnetic forces

⊛ the observations above, when combined with constancy of the speed of light, lead to very strange consequences.

Question: Now there are 3 things in the universe. You are at rest, your friend moves toward you at a speed close to c, and light travels toward you from opposite direction. What do you and your friend see?



⊛ You measure that your friend and light approach each other with speed roughly 2c.

⊛ Your friend measures light to approach him at speed c (remember, speed of light is constant).

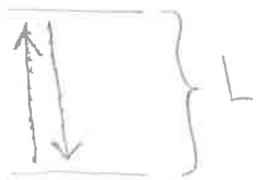
This can only happen if you and your friend measure space and/or time differently!

Let's perform a few well controlled experiments to determine how you and your friend experience reality differently.

Time Dilation

At the moment your friend passes you, she bounces light up and down between 2 mirrors

Her frame:

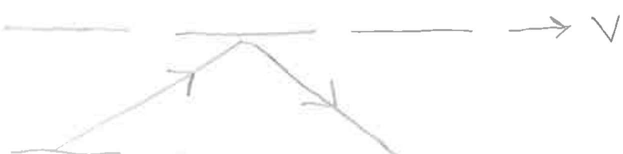


$$y = v t_1$$

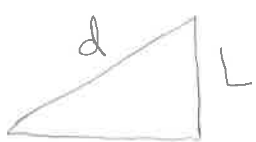
$$2L = c t_1$$

$$\boxed{\frac{2L}{c} = t_1}$$

Your frame:



Light takes longer path, but since speed of light is constant, it must also take a longer time t_2.



$x = v\left(\frac{1}{2}t_2\right)$

$$d^2 = L^2 + \frac{1}{4}v^2 t_2^2$$

$\Rightarrow$ Light travels total distance $2d$:

$$2d = c t_2$$

$$t_2 = \frac{2\sqrt{L^2 + \frac{1}{4}v^2 t_2^2}}{c}$$

$$\frac{1}{4}c^2 t_2^2 = L^2 + \frac{1}{4}v^2 t_2^2$$

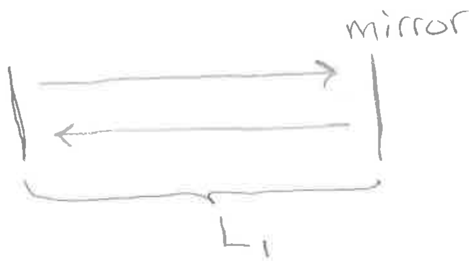
Time Dilation $\boxed{t_2 = \frac{2L}{\sqrt{c^2 - v^2}} = \frac{t_1}{\sqrt{1 - v^2/c^2}}}$

⊛ You see your friend's clock ticking slower ($t_1 < t_2$)!

Length Contraction

Suppose instead that you bounce light horizontally.

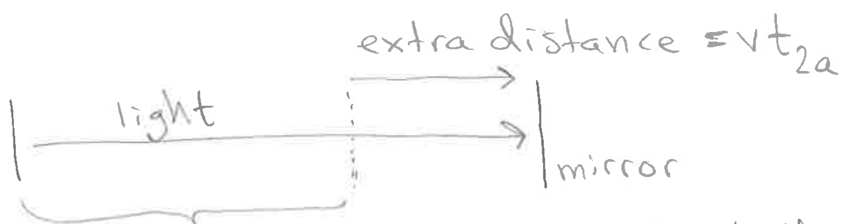
Friend frame:



$2L_1 = ct_1$

$$\boxed{t_1 = \frac{2L_1}{c}}$$

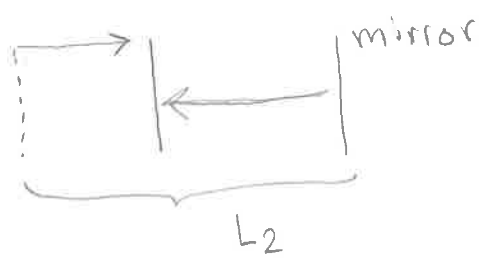
Your frame:



L_2 original length (might be different than L_1)

Light travels distance $L_2 + vt_{2a} = ct_{2a}$ during first half of trip $\Rightarrow t_{2a} = \dfrac{L_2}{c-v}$.

Returning:



During return trip, light travels shorter distance:

$$L_2 - vt_{2b} = ct_{2b}$$

$$\Rightarrow t_{2b} = \frac{L_2}{v+c}.$$

Total time measured by you :

$$t_2 = t_{2a} + t_{2b} = L_2 \left(\frac{1}{c-v} + \frac{1}{c+v} \right)$$

$$= L_2 \frac{2c}{c^2 - v^2}$$

But since $t_2 = \dfrac{t_1}{\sqrt{1 - v^2/c^2}}$,

$$\frac{t_1}{\sqrt{1 - v^2/c^2}} = L_2 \frac{2c}{c^2 - v^2}$$

$$\frac{2L_1}{c\sqrt{1 - v^2/c^2}} = \frac{2cL_2}{c^2 - v^2}$$

$$\boxed{L_2 = \sqrt{1 - v^2/c^2}\, L_1}$$ Length contraction

* You now see all objects to be shorter along direction of motion !

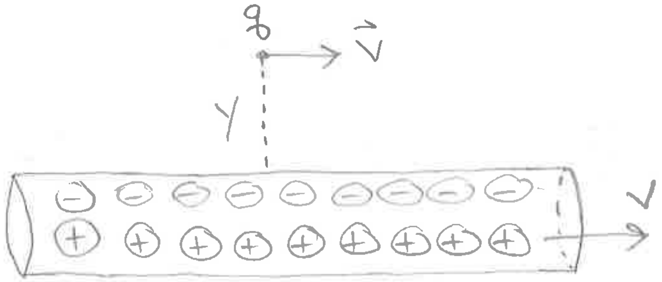
* These effects are typically small if v is much less than c.

People often say relativity only applies when objects move close to the speed of light.

They are wrong !

(*) Relativity is essential just to have a consistent theory of electromagnetism.

Example: Consider a point charge q that moves parallel to a long current carrying wire. For simplicity assume that q and the mobile charges in wire move with same speed v.



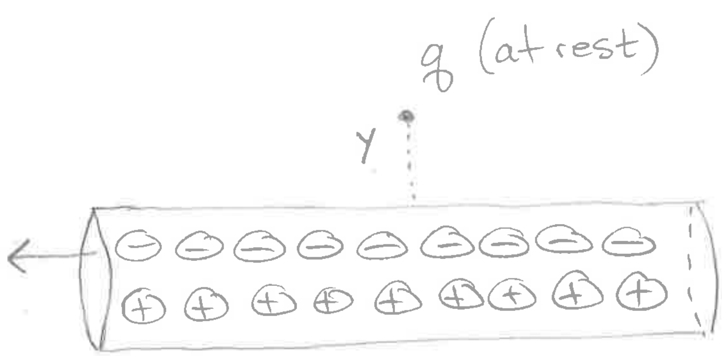
Protons move right with velocity $\vec{v}$, while electrons are stationary

In this frame there is a magnetic force $\vec{F}_B = q\vec{v}\times\vec{B}$.

$$\vec{B} = \frac{\mu_0 i}{2\pi y}\odot \Rightarrow \boxed{\vec{F}_B = \frac{-\mu_0 i}{2\pi y} q v (\hat{i}_y)}$$

(neglect current in other parts of wire that are far away)

(*) If we now move to reference frame in which q is at rest, there seems to be no reason for q to move down:



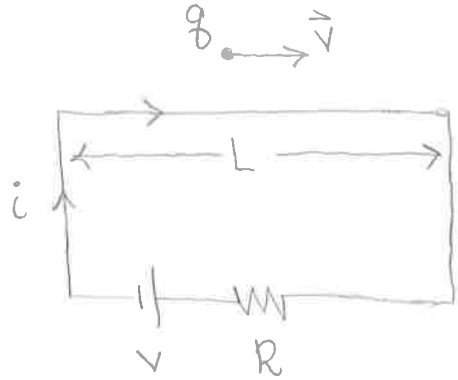
Now protons are at rest and electrons move left with speed v.

(1) No magnetic force on q since it is at rest

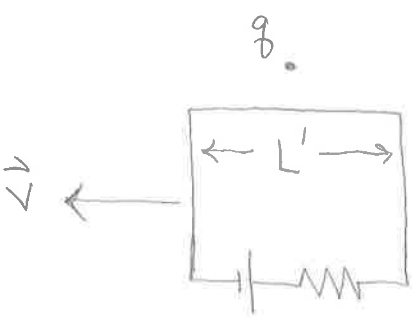
(2) No induced $\vec{E}$ field from Faraday's Law since $\vec{B}$ is not changing in time.

✻ The only way out of this problem is special relativity.

Let's think about the charge density in the two frames:



In first reference frame, the linear charge density $\boxed{\lambda = \frac{Q}{L}}$ must be the same everywhere from symmetry.



In second reference frame, the length is contracted to

$$\boxed{L' = \sqrt{1 - v^2/c^2}\; L}$$

Since the ⊖ charges do not move in the wire:

$$\lambda'_{-} = \frac{-Q}{L'} = \frac{-1}{\sqrt{1-v^2/c^2}} \frac{Q}{L} = \frac{-\lambda}{\sqrt{1-v^2/c^2}}$$

What about ⊕ charges? Just the opposite!

The spacing between ⊕ charges in second frame must be larger because now they are at "rest":

$$\boxed{\lambda'_{+} = \sqrt{1-v^2/c^2}\; \lambda}$$

Total charge density in new frame:

$$\lambda' = \lambda'_+ + \lambda'_- = \lambda\left(\left(1-\frac{v^2}{c^2}\right)^{+1/2} - \left(1-\frac{v^2}{c^2}\right)^{-1/2}\right)$$

Recall that for small x, $(1+x)^\alpha \approx 1+\alpha x + \cdots$

$$\Rightarrow \lambda' = \lambda\left(1-\frac{1}{2}\frac{v^2}{c^2} - \left(1+\frac{1}{2}\frac{v^2}{c^2}\right)\right) = \boxed{-\lambda\left(\frac{v^2}{c^2}\right)}$$

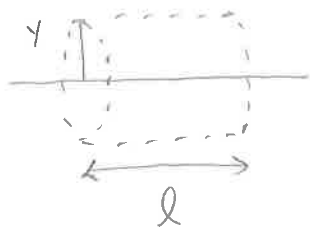
(Note: average drift speed of electrons in real metals is $v \approx 1$ mm/s, so yes $\frac{v}{c}$ is very small)

(*) From relativity there is a net negative charge density that now produces an $\vec{E}$ field that attracts q downward!

There's more ...

Gaussian surface

From Gauss's Law



$$\oint \vec{E}\cdot dA = \frac{q_{enc}}{\varepsilon_0}$$

$$E(2\pi y \ell) = \frac{1}{\varepsilon_0}(\lambda' \ell) = \frac{1}{\varepsilon_0}\left(-\frac{v^2}{c^2}\lambda\right)\ell$$

$$\Rightarrow \boxed{E = \frac{-1}{2\pi\varepsilon_0}\frac{\lambda}{y}\frac{v^2}{c^2}}$$

But note that

$$\lambda v = \left(\frac{dq}{dx}\right)\left(\frac{dx}{dt}\right) = \frac{dq}{dt} = i$$

$$\Rightarrow \quad E = \frac{-1}{2\pi\varepsilon_0}\frac{\lambda}{y}\frac{v^2}{c^2} = \frac{-1}{2\pi\varepsilon_0}\frac{i\,v}{y\,c^2}.$$

But $\frac{1}{c^2} = \mu_0\varepsilon_0$

$$\Rightarrow \quad E = \frac{-1}{2\pi\varepsilon_0}\frac{i\,v}{y}\mu_0\varepsilon_0 = \left(\frac{-\mu_0 i}{2\pi\varepsilon_0 y}\right)v$$

$$\Rightarrow \quad F_E = qE$$

$$\boxed{\vec{F}_E = \frac{-\mu_0 i}{2\pi y}qv\,(\hat{i}_y)}$$

But what was the magnetic force $\vec{F}_B$ in the original frame?

$$\boxed{\vec{F}_B = \frac{-\mu_0 i}{2\pi y}qv\,(\hat{i}_y)}$$