PHYS 207 Electricity and Magnetism

13. Maxwell's Equations and Electromagnetic Waves

13.1) Displacement Currents

13.2) Maxwell's equations and free-space solutions

13.3) Properties of electromagnetic waves

13.1 Displacement Currents

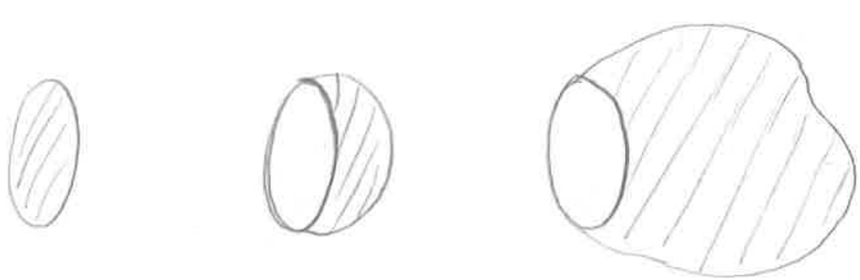
Recall Ampere's Law:

$$\oint_C \vec{B}\cdot d\vec{\ell} = \mu_0 i_{enc}$$

The current enclosed i_{enc} is the current that passes through the surface defined by loop C.

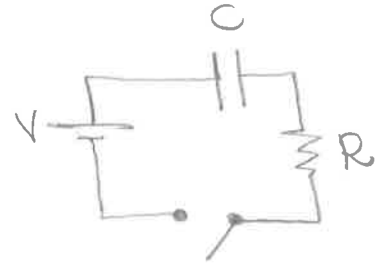
(*) Maxwell noticed a "loop-hole" in this law:

$$\boxed{\text{There are infinitely many surfaces with C as their boundary!}}$$



Why should we care??

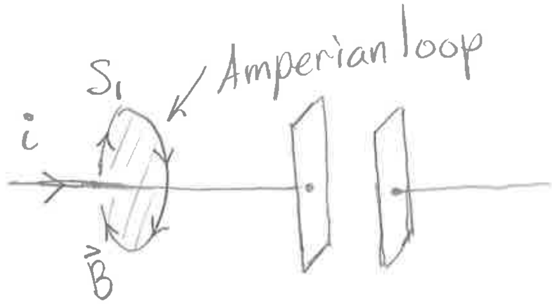
Reason: Suppose we have a capacitor that is charging in a circuit.



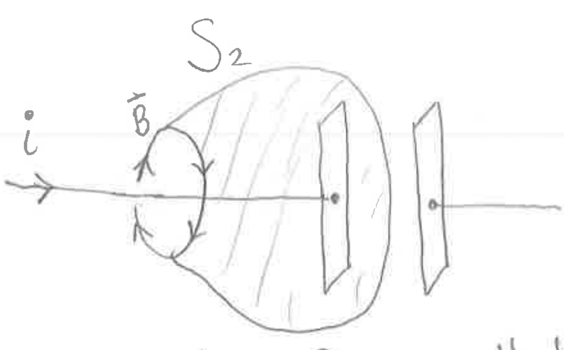
We know that $Q(t) = CV\left(1 - e^{-t/RC}\right)$ and

$$i(t) = \frac{V}{R} e^{-t/RC}$$

Look at the current-carrying wire just before the capacitor



$$\oint \vec{B} \cdot d\vec{l} = \mu_0 i$$



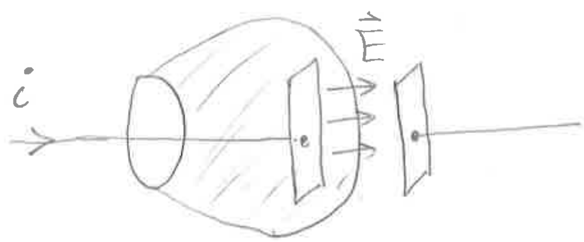
Extend surface so that no current is enclosed.

$$\oint \vec{B} \cdot d\vec{l} \neq \mu_0 i_{enc}$$

$$(i_{enc} = 0)$$

⊛ Maxwell realized that even though there is no current passing through this deformed surface S_2, there is an electric flux through S_2

$$\phi_E = \int \vec{E} \cdot d\vec{A}$$



⊛ But after a very long time $\vec{B}$ goes to 0:

$$i \to 0 \quad \left(\oint \vec{B} \cdot d\vec{l} = \mu_0 i_{enc} \to 0 \right)$$

$$Q \to CV \;\Rightarrow\; \phi_E = \int \vec{E} \cdot d\vec{A} \neq 0$$

This led Maxwell to guess that either current passing through a surface (i_{enc}) or a changing electric flux through a surface produces a $\vec{B}$ field around the surface boundary

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \varepsilon_0 \frac{d\phi_E}{dt}$$

The quantity $\varepsilon_0 \dfrac{d\phi_E}{dt}$ has units of current

$$\Rightarrow \quad i_D = \varepsilon_0 \frac{d\phi_E}{dt} \quad \text{"displacement current"}$$

New Ampere Law:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \left(i_{enc} + \varepsilon_0 \frac{d\phi_E}{dt} \right)$$

Another motivation for Maxwell was symmetry:

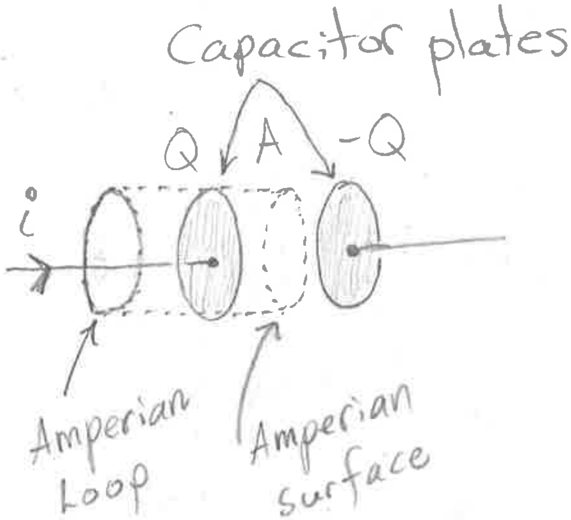
$$\mathcal{E}_{ind} = \oint \vec{E} \cdot d\vec{l} = -\frac{d\phi_B}{dt} \quad \text{(Faraday's Law)}$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \varepsilon_0 \frac{d\phi_E}{dt} \quad \text{(Maxwell-Ampere Law)}$$

⊛ This simple observation will soon have a profound consequence : <u>electromagnetic waves</u>

But first let's look at a rather simple application :

<u>Magnetic field inside a charging capacitor</u>



$$Q = C\Delta V$$
$$= C(Ed)$$
$$= \left(\varepsilon_0 \frac{A}{d}\right)(Ed)$$
$$= \varepsilon_0 A E = \varepsilon_0 \phi_E$$

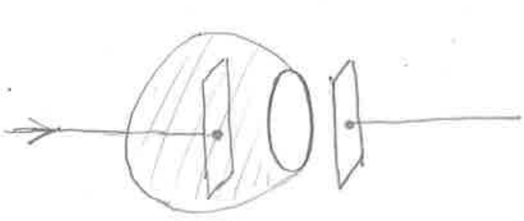
$$\Rightarrow \quad Q = \varepsilon_0 \phi_E$$

$$i = \frac{dQ}{dt} = \varepsilon_0 \frac{d\phi_E}{dt}$$

This quantity is exactly equal to the current i!

But $\frac{dQ}{dt}$ is the current in the wire that is charging the capacitor.

Therefore $i_D \equiv \varepsilon_0 \frac{d\phi_E}{dt} = i$ (this must be so because we can just deform our surface again)



⊛ There must be a $\vec{B}$ field inside the charging capacitor

Let's summarize our Knowledge of how electric and magnetic fields are produced.
Together with the fundamental relationships between fields and forces ($\vec{F} = \vec{E}q$ and $\vec{F} = q\vec{v} \times \vec{B}$), that's all there is.

13.2) Maxwell's Equations and Electromagnetic Waves

(i) $\oint \vec{E} \cdot d\vec{A} = \dfrac{q}{\varepsilon_0}$ (iii) $\oint \vec{B} \cdot d\vec{A} = 0$

(ii) $\oint \vec{E} \cdot d\vec{l} = \dfrac{-d\phi_B}{dt}$ (iv) $\oint \vec{B} \cdot d\vec{l} = \mu_0 i + \mu_0 \varepsilon_0 \dfrac{d\phi_E}{dt}$

⊛ When there are no "sources" (i.e., charges or currents), Maxwell's equations reduce to

$$\left.\begin{aligned} &\oint \vec{E} \cdot d\vec{A} = 0 \qquad \oint \vec{B} \cdot d\vec{A} = 0 \\ &\oint \vec{E} \cdot d\vec{l} = -\frac{d\phi_B}{dt} \qquad \oint \vec{B} \cdot d\vec{l} = \mu_0 \varepsilon_0 \frac{d\phi_E}{dt} \end{aligned}\right\} \text{Very Symmetric}$$

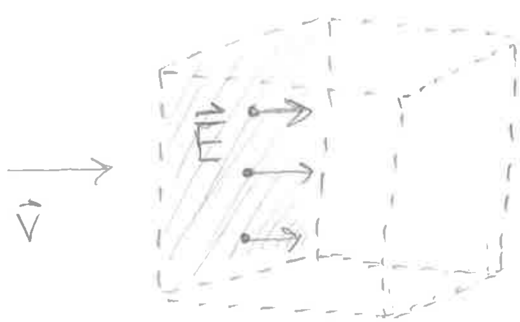
⊛ Is it possible to find $\vec{E}$ and $\vec{B}$ field configurations that satisfy these reduced equations?

Ans: Yes, and there are several different types of EM waves

Example: "Plane waves" have uniform $\vec{E}$ and $\vec{B}$ fields in a plane that is perpendicular to the direction of motion

First question: Can $\vec{E}$ or $\vec{B}$ point in the direction of propagation?

Ans: No, that would violate $\oint \vec{E} \cdot d\vec{A} = 0 = \int \vec{B} \cdot d\vec{A}$



As EM wave enters our Gaussian surface, the total flux will be nonzero.

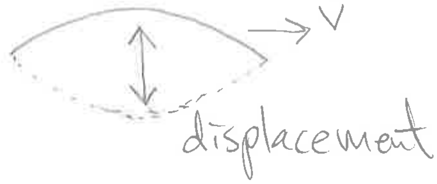
⊛ this is somewhat natural: many waves behave like this

⊛ Disturbance perpendicular to velocity
$\Rightarrow$ "transverse wave"

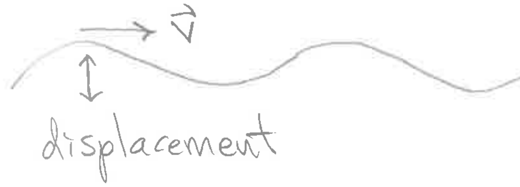
⊛ Disturbance parallel to velocity $\Rightarrow$ "longitudinal wave"

Examples:

Guitar strings

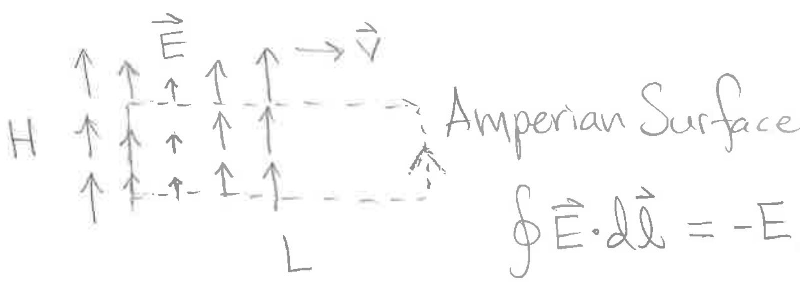


Water waves



Sound waves

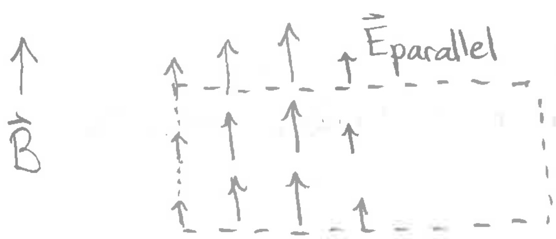
(*) How are $\vec{E}$ and $\vec{B}$ oriented relative to each other?



$$\oint \vec{E} \cdot d\vec{l} = -EH = -\frac{d\phi_B}{dt}$$

$\Rightarrow$ $\vec{B}$ must point out of page

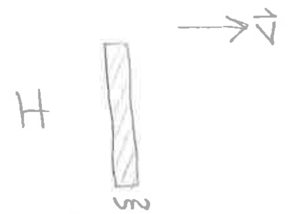
In fact, $\vec{E}$ and $\vec{B}$ must be completely perpendicular:



Loop that completely parallel to $\vec{B}$

$$\oint \vec{E}_p \cdot d\vec{l} \neq 0 \quad \text{but} \quad \frac{d\phi_B}{dt} = 0.$$

The quantity $\frac{d\phi_B}{dt}$ depends on the speed of the wave:



$(dx = v\,dt)$

$$\frac{d\phi_B}{dt} = EH$$

$$d\phi_B = EH \cdot dt$$

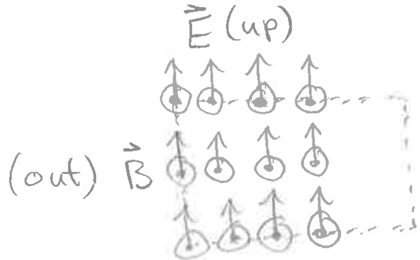
$$B\,dA = B(H\,dx) = BHv\,dt = EH\,dt$$

$$\Rightarrow Bv = E \Rightarrow \boxed{B = \frac{1}{v}E}$$

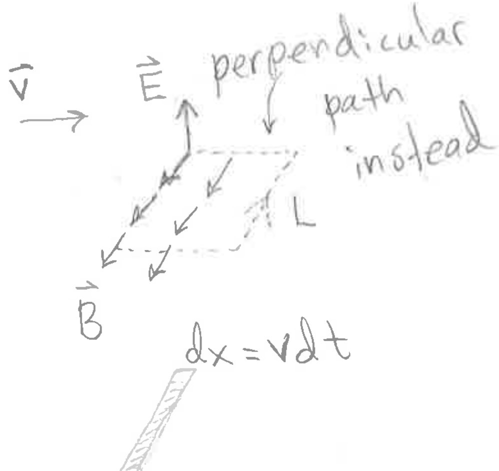
(*) $\vec{E}$ and $\vec{B}$ field strengths related by velocity!

But we are not finished...

Same analysis with $\vec{B}$ field but use different path:



This path would give $\oint \vec{E} \cdot d\vec{l} = 0$



$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \varepsilon_0 \frac{d\phi_E}{dt}$$

$$BL = \mu_0 \varepsilon \left(EL \frac{dx}{dt}\right)$$

$$\Rightarrow \boxed{E = \frac{\mu_0 \varepsilon_0}{v} B}$$

(a) $Bv = E$ and (b) $B = \mu_0 \varepsilon_0 E v$

$$\Rightarrow \quad B = \mu_0 \varepsilon_0 B v^2 \Rightarrow v^2 = \frac{1}{\mu_0 \varepsilon_0}$$

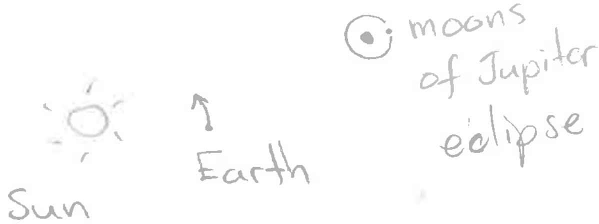
$$\Rightarrow \quad \boxed{v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}}$$

(*) But we know μ_0 and ε_0 from experiments:

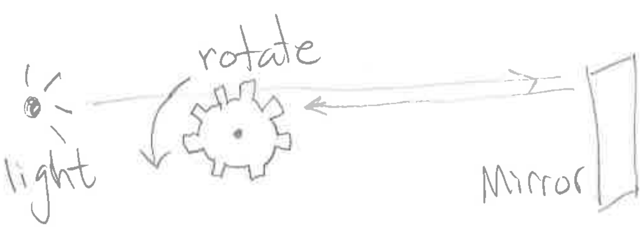
$$\mu_0 = 4\pi \times 10^{-7} \frac{N}{A^2} \qquad \varepsilon_0 = 8.85 \times 10^{-12} \frac{C^2}{N \cdot m^2}$$

$$\Rightarrow \quad \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8 \, m/s$$ (hmm, exactly the speed of light!)

(*) At the time of Maxwell, the speed of light was well known:



$c \approx 2 \times 10^8 \, m/s$ (~1650)



$c \approx 3 \times 10^8 \, m/s$ (~1850)

This led Maxwell to conclude that light is an electromagnetic wave.

(*) Apparently, there is only one possible speed at which light can travel: $v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 8 \times 10^8 \, m/s$!

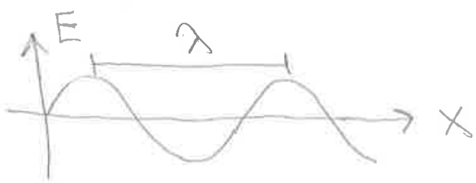
13.3) Properties of electromagnetic waves

Two ways to think about waves:

(a) Fix time (take "snapshot" of position dependence)

(b) Fix position (study time dependence)

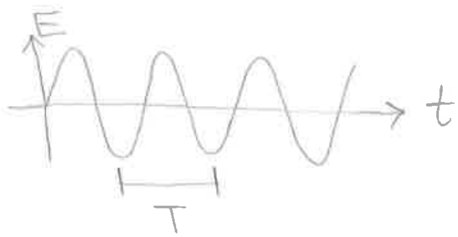
(*) Fix time:



"Wavelength" λ

= peak-to-peak distance

(*) Fix position:



"Period" T

= peak-to-peak time

(*) For general waves, λ and T are independent but determine the velocity

$$\boxed{v = \frac{\lambda}{T}}$$

⊛ Position and time dependence

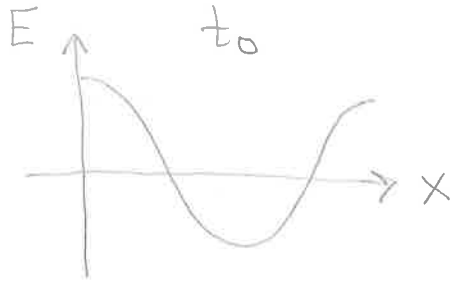
$$E(x,t) = E_0 \cos\left(\frac{2\pi}{\lambda}x - \frac{2\pi}{T}t\right)$$

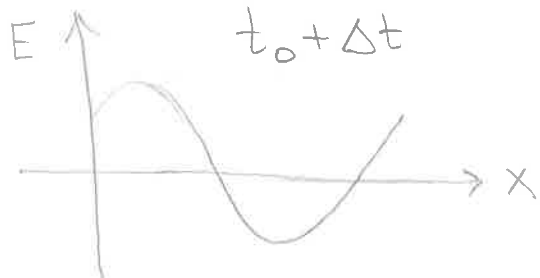
$$E(x+\lambda,t) = E_0 \cos\left(\frac{2\pi}{\lambda}x + 2\pi - \frac{2\pi}{T}t\right)$$

$$\boxed{\begin{array}{c} \frac{2\pi}{\lambda} \equiv K \\ \frac{2\pi}{T} \equiv \omega \end{array}}$$

In total : $E(x,t) = E_0 \cos(Kx - \omega t)$

why - sign??





$E_0 \cos(\underbrace{Kx - \omega t_0})$

peak at $x_p = \frac{\omega t_0}{K}$

$E_0 \cos(\underbrace{Kx - \omega(t_0 + \Delta t)})$

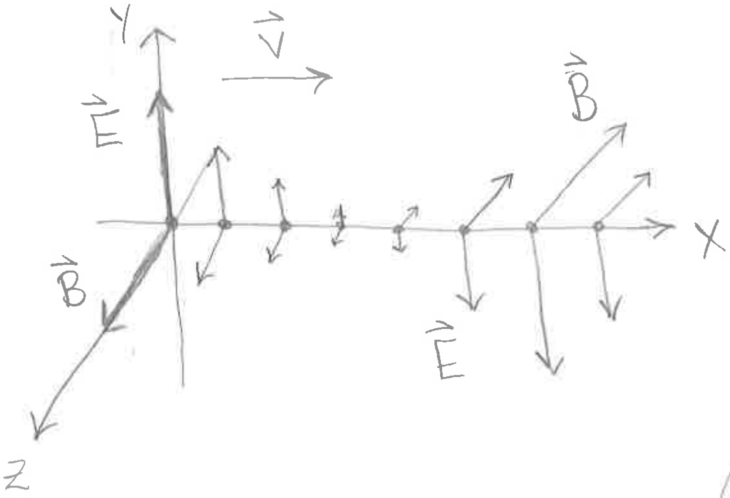
peak at $x_p = \frac{\omega(t_0 + \Delta t)}{K}$

New peak slightly to the right

⊛ Right-moving wave : $E_0 \cos(Kx - \omega t)$

Left-moving wave : $E_0 \cos(Kx + \omega t)$

⊛ Exact same for oscillating $\vec{B}$ fields



Travel in +x direction

$$\vec{E}(x,t) = E_0 \cos(kx - \omega t)\,\hat{i}_y$$

Direction of travel

Direction of $\vec{E}$ field

$$\vec{B}(x,t) = \frac{E_0}{c}\cos(kx - \omega t)\,\hat{i}_z$$

Direction of $\vec{B}$

Same as for $\vec{E}$

$|\vec{E}| = c|\vec{B}|$

Same as for $\vec{E}$