12.1) Reference frames with constant velocity
12.2) Accelerating reference frames
12.3) Fictional forces

12.1) Reference frames with constant velocity

- Until now we have been able to choose the location of the origin and orientation of the x, y, z axes, but we have always implicitly assumed they were stationary.

- Once the origin is chosen, all vectors $\{\vec{r}, \vec{v}, \vec{a}, \vec{F}, \vec{L}, \vec{\tau}, \ldots\}$ can be defined and physical laws relating them can be derived $\left\{\vec{F} = m\vec{a},\ \vec{L} = \vec{r} \times \vec{p},\ \vec{\tau} = \frac{d\vec{L}}{dt}, \ldots\right\}$.

- You might ask yourself: "What if I had chosen a moving origin or coordinate system? Would the fundamental laws of physics have to be modified?"

(i) Coordinate system moving with constant velocity $\longrightarrow$ Laws of physics unchanged

(ii) Accelerating coordinate system $\longrightarrow$ Laws of physics change

<u>Example</u>: Suppose you are sitting on a train which travels at a constant velocity $\vec{v} = v_0 \hat{i}_x$. At time $t=0$ you pass person A outside. Define two coordinate systems: $\mathcal{O}$ is centered on person A and $\mathcal{O}'$ is centered on you in the train:

(a) Your position is given by:

$$\mathcal{O}: \vec{a} = 0, \; \vec{v} = v_0 \hat{i}_x, \; \vec{x} = v_0 t \, \hat{i}_x$$

$$\mathcal{O}': \vec{a}' = 0, \; \vec{v}' = 0, \; \vec{x}' = 0$$

(b) Throw an apple at horizontal speed v_a to your friend on opposite end of train car. Apple's trajectory is given by:

$$\mathcal{O}: \vec{a} = -g\hat{i}_y, \; x = (v_0 + v_a)t, \; y = -\tfrac{1}{2} g t^2$$

$$\mathcal{O}': \vec{a}' = -g\hat{i}_y, \; x' = v_a t, \; y' = -\tfrac{1}{2} g t^2$$

(*) In both reference frames, the normal laws of physics can be applied. In particular, we used:

$$F_x = ma_x = 0 \Rightarrow a_x = 0 \quad \text{and}$$
$$F_y = ma_y = -mg \Rightarrow a_y = -g.$$

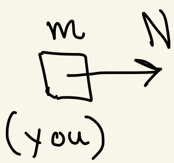
(*) Constant velocity reference frames introduce no important complications.

12.2) Accelerating reference frames

Example: Now suppose that the train starts from rest and has acceleration $a_x = \alpha$. Your trajectory in the two frames is:

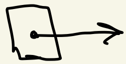
$$\mathcal{O}: a_x = \alpha, \; v = \alpha t, \; x = \tfrac{1}{2}\alpha t^2$$
$$\mathcal{O}': a'_x = 0, \; v' = 0, \; x' = 0$$

(*) In $\mathcal{O}$ we see that Newton's 2nd law holds:



Normal force from seat causes you to accelerate

$$N = ma = m\alpha$$

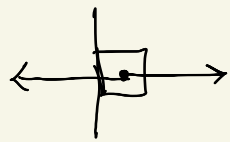
* In $\mathcal{O}'$ there is the same normal force N, but you don't accelerate 

$\vec{F} = m\vec{a}$ is no longer valid!

* In an accelerating frame, you must introduce fictional forces in order to apply Newton's 2nd Law.

In the previous train example, it would "feel" to you that some mystery force F_m is is pushing you against your seat:

$$F_{total} = N - F_m = 0$$

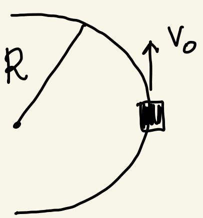


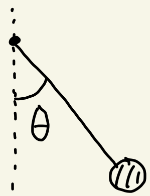
But $N = m\alpha$ (from above)

$$\Rightarrow \vec{F}_m = -m\alpha\,\hat{i}_x$$

* Note that $\vec{F}_m$ is opposite to the direction of acceleration. This is a general feature of fictitious forces in accelerating reference frames.

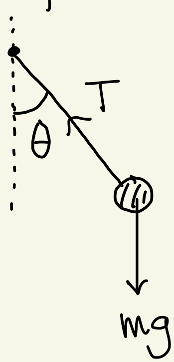
<u>Example</u>: Suppose you hang a ball of mass m from the ceiling of your train that travels at constant speed v_0 around a circular track of radius R. What will be the tension in the string and at what angle will it hang?





In your reference frame, it appears there is a mysterious force pushing ball to right.

In reality, there must be a centripetal force $\vec{F}_c = -m\omega^2 \hat{\imath}_r = -m\left(\frac{v_0}{R}\right)^2 \hat{\imath}_r$ pointing inward toward the center of circle. This is produced by the physical tension force



x) $T\sin\theta = ma = m\left(\frac{v_0}{R}\right)^2$

y) $T\cos\theta - mg = 0$

$\Rightarrow T\cos\theta = mg$

x) $T\sin\theta = m\left(\frac{v_0}{R}\right)^2$

y) $T\cos\theta = mg$

Divide equations: $\tan\theta = \frac{v_0^2}{gR^2}$

Sum squares of both equations:

$$T^2\sin^2\theta + T^2\cos^2\theta = m^2\left(\frac{v_0}{R}\right)^4 + m^2g^2$$

$$\Rightarrow T = m\sqrt{\left(\frac{v_0}{R}\right)^4 + g^2}$$

In your reference frame, the fictitious force to the right is called the "centrifugal force".

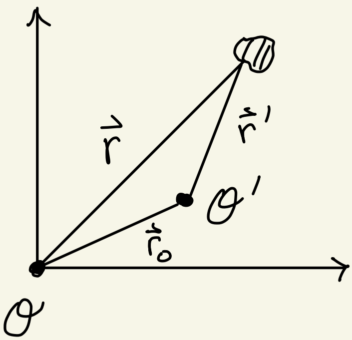
12.3) Fictional forces

General derivation:

(1) Define $\vec{r}_0$ as the location of the $\mathcal{O}'$ (accelerating) origin in the $\mathcal{O}$ frame.

(2) The position of some object is $\vec{r}$ in the $\mathcal{O}$ frame and $\vec{r}\,'$ in the $\mathcal{O}'$ frame.

(3) Then



$$\vec{r} = \vec{r}_0 + \vec{r}\,'$$

$$\frac{d\vec{r}}{dt} = \frac{d\vec{r}_0}{dt} + \frac{d\vec{r}\,'}{dt}$$

$$\frac{d^2\vec{r}}{dt^2} = \frac{d^2\vec{r}_0}{dt^2} + \frac{d^2\vec{r}\,'}{dt^2}$$

$$\vec{a} = \frac{d^2\vec{r}_0}{dt^2} + \vec{a}\,'$$

$$\frac{\vec{F}}{m} = \frac{d^2\vec{r}_0}{dt^2} + \vec{a}\,'$$

$$\Rightarrow \boxed{\underbrace{\vec{F}}_{\text{Real forces}} - \underbrace{m\frac{d^2\vec{r}_0}{dt^2}}_{\text{Fictional forces}} = \underbrace{m\vec{a}\,'}_{\text{Acceleration in moving frame.}}}$$

(*) Fictional forces always act opposite to true acceleration of $\mathcal{O}'$ in $\mathcal{O}$ "inertial frame".

Case 1 Suppose O' system rotates with constant angular velocity ω at distance d from origin O. Then

$$F_r - (-md\omega^2) = ma_{r'}$$

$$\Rightarrow \quad F_r + md\omega^2 = ma_{r'}$$

Centrifugal force radially outward

Case 2 Suppose O' system rotates with constant angular velocity ω and has constant radial velocity $v_r = \frac{dr_0}{dt}$. then

$$F_r + mr_0\omega^2 = ma_{r'}$$

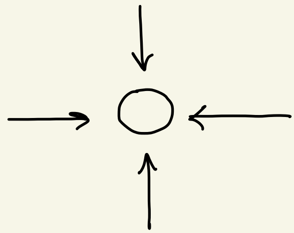
$$F_\theta - m\Big(\underbrace{r_0\alpha}_{0} + 2\omega\frac{dr_0}{dt}\Big) = ma_{\theta'}$$

$$\Rightarrow \quad F_\theta - 2m\omega v_r = ma_{\theta'}$$

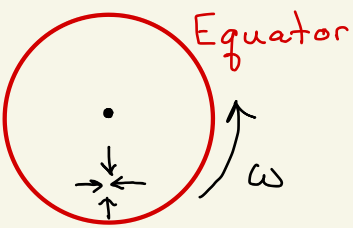
"Coriolis force" acts tangentially.

<u>Example:</u> Hurricanes in the northern hemisphere rotate counterclockwise, and those in the southern hemisphere rotate clockwise.

Eye of hurricane is a low-pressure area that sucks air in.



In northern hemisphere, top air actually flows <u>radially</u> <u>outward</u>



(top view)

air

Coriolis force

$-2m\omega v_r < 0 \quad (v_r > 0)$

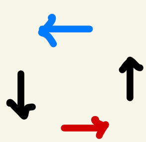
F_θ

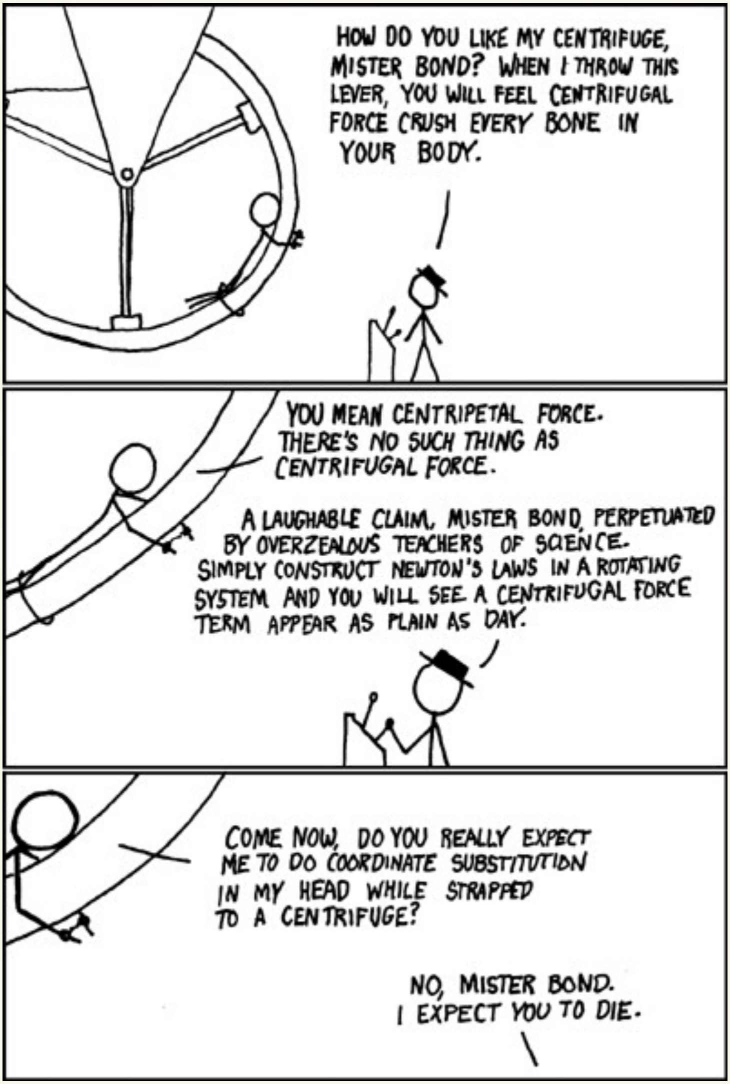
F_θ Coriolis force

$-2m\omega v_r > 0 \quad (v_r < 0)$

air

Result is counterclockwise rotation




HOW DO YOU LIKE MY CENTRIFUGE, MISTER BOND? WHEN I THROW THIS LEVER, YOU WILL FEEL CENTRIFUGAL FORCE CRUSH EVERY BONE IN YOUR BODY.
YOU MEAN CENTRIPETAL FORCE. THERE'S NO SUCH THING AS CENTRIFUGAL FORCE.
A LAUGHABLE CLAIM, MISTER BOND, PERPETUATED BY OVERZEALOUS TEACHERS OF SCIENCE. SIMPLY CONSTRUCT NEWTON'S LAWS IN A ROTATING SYSTEM AND YOU WILL SEE A CENTRIFUGAL FORCE TERM APPEAR AS PLAIN AS DAY.
COME NOW, DO YOU REALLY EXPECT ME TO DO COORDINATE SUBSTITUTION IN MY HEAD WHILE STRAPPED TO A CENTRIFUGE?
NO, MISTER BOND. I EXPECT YOU TO DIE.