PHYS 207 Electricity and Magnetism

12. Time-Dependent Circuits

12.1) Inductors

So far we have considered batteries, resistors, and capacitors in building up electrical circuits.

Generically, (1) batteries supply energy

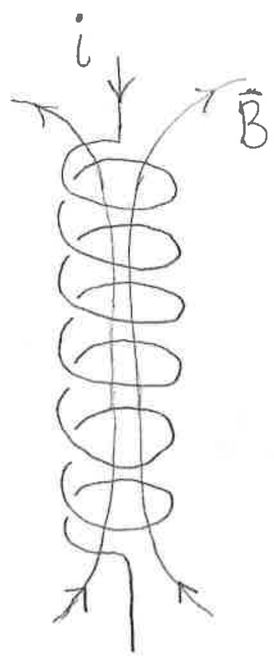
(2) resistors dissipate energy

(3) capacitors store energy in the electric field

(*) Inductors are a type of circuit element that can store magnetic field energy (but their properties will be very different than those of capacitors.

Example: A coiled wire is the simplest example of an inductor:

(*) <u>1st property</u>: when current flows through the wire, it will generate a strong magnetic field (and magnetic flux) through the coil.



Here there are 6 loops

In general, it is too difficult to actually compute ϕ_B exactly.

But since $\vec{B}$ is proportional to i $\left(d\vec{B} = \frac{\mu_0 i}{4\pi} \frac{d\hat{l} \times \hat{r}}{r^2}\right)$, the magnetic flux through the coil is also proportional to i:

$$\boxed{\phi_B = L i}$$

The proportionality constant L is called the "<u>inductance</u>" of the coil.

(*) <u>Important point</u>: Do <u>not</u> think of the coil as a 3-dimensional volume (remember $\oint \vec{B} \cdot d\vec{A} = 0$). Instead, ϕ_B is the total flux computed as the sum through each of the individual loops in the coil. In the figure above, $\phi_B \sim 6BA$, where A is the loop area.

(*) 2nd property: The inductance L depends only on the geometry of the coiled wire (just like capacitance only depended on geometry of the two conductors).

This is because the pattern of the magnetic field depends only on the geometry of the coil. Changing the current doesn't change the $\vec{B}$ field pattern, only its strength.

(*) Third property (maybe most important): Inductors oppose changes in the current. From Faraday's Law

$$\mathcal{E}_{ind} = -\frac{d\phi_B}{dt}$$

$$\mathcal{E}_{ind} = -\frac{d}{dt}(Li)$$

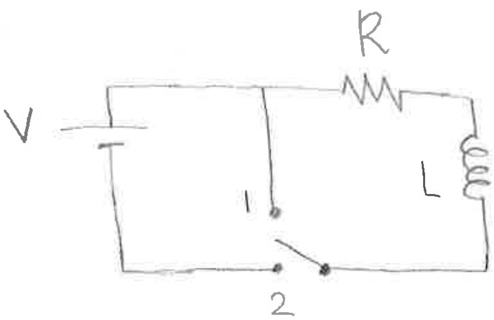
$$\boxed{\mathcal{E}_{ind} = -L\frac{di}{dt}}$$

(*) This expression can be used in Kirchhoff's Loop Rule when path is in direction of the current.

Note that the minus sign in the above expression indicates that $\mathcal{E}_{ind}$ opposes changes in current.

12.2) RL circuits

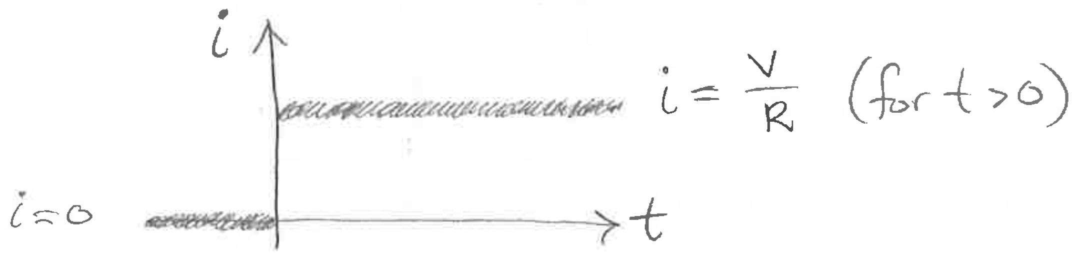
Let's see what happens when we place an inductor in series with a battery and capacitor.



Inductor symbol is a coil.

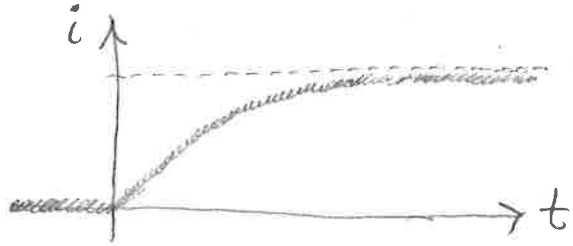
At time $t=0$ we flip the open switch to position 2.

Without the inductor, there would be a nearly discontinuous change in the current:

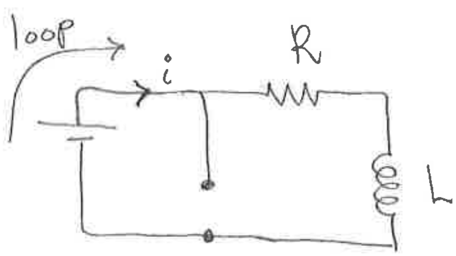


(*) this would produce an infinite value of $\frac{di}{dt}$ and therefore an infinite induced opposing electromotive force in the inductor: $\mathcal{E}_{ind} = -L\frac{di}{dt} = -\infty$.

⊛ Instead, the inductor smooths out the increase in the current:



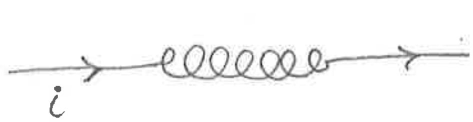
We can prove this using Kirchhoff's Loop Rule:



Outer Loop Clockwise: $V - iR - L\frac{di}{dt} = 0$

Note that $\frac{di}{dt} > 0$ and therefore $-L\frac{di}{dt} < 0$, which means it opposes the battery's own emf.

⊛ Important: Inductors in Kirchhoff's Loop Rule are simple



Current flows to right

$-L\frac{di}{dt}$ (path in direction of i)

$+L\frac{di}{dt}$ (path opposite of i)

How to solve $V - iR - L\frac{di}{dt} = 0$?

This is a differential equation (can be challenging to solve in general).

Here we use technique of "separation of variables" (arrange so that i and t are on opposite sides of equation and integrate)

$$V - iR = L\frac{di}{dt} \Rightarrow \frac{dt}{L} = \frac{di}{V - iR}$$

$$\Rightarrow \frac{1}{L}\int_0^t dt = \int_{i(0)}^{i(t)} \frac{di}{V - Ri}$$

$$\Rightarrow \frac{t}{L} = -\frac{1}{R} \ln(V - Ri) \Big|_{i(0)}^{i(t)}$$

from chain rule

$i(0) = 0$

$$\Rightarrow \frac{t}{L} = -\frac{1}{R}\left[\ln(V - Ri) - \ln(V - 0)\right]$$

$$\Rightarrow -\frac{R}{L}t = \ln\left(1 - \frac{R}{V}i\right)$$

$$\Rightarrow e^{-\frac{R}{L}t} = 1 - \frac{R}{V}i$$

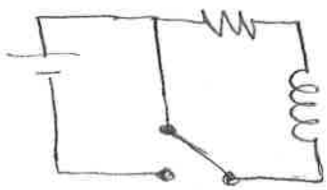
$$\Rightarrow i(t) = \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

$$\left.\begin{aligned} &\text{At } t=0,\ i(0) = \frac{V}{R}\left(1-e^{0}\right) = 0 \\ &\text{At } t\to\infty,\ i(\infty) = \frac{V}{R}\left(1-e^{-\infty}\right) = \frac{V}{R} \end{aligned}\right\}$$ As we expected.

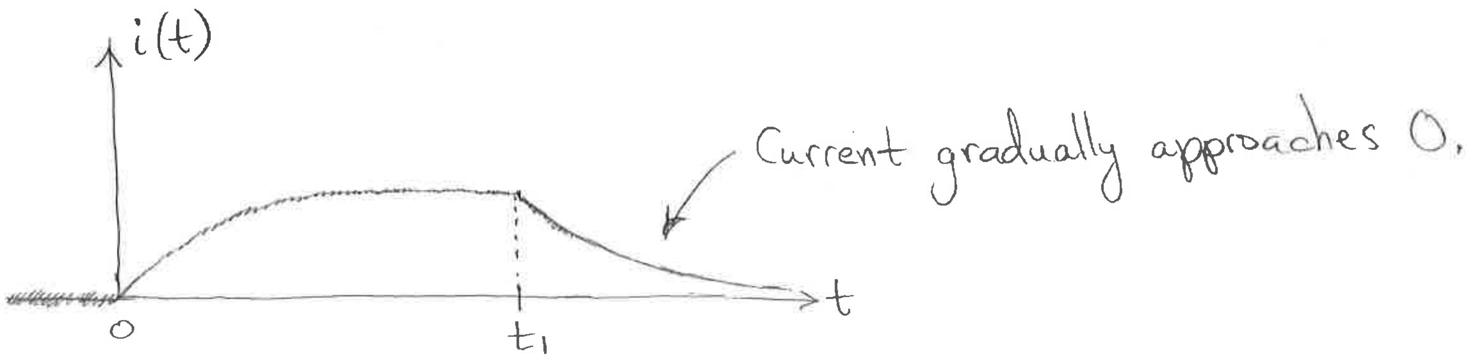
(*) Note that after a long time, inductors behave just like conducting wires:

$$\mathcal{E}_{ind} = -L\frac{di}{dt} \to 0 \text{ since } \frac{di}{dt} \to 0.$$

Question: what happens if we now flip the switch to position labeled 1 at time t_1?



Key Point: Inductors prevent discontinuous changes in current since then $\mathcal{E}_{ind} = -L\frac{di}{dt} \to \infty$.



(*) The battery is disconnected, so where does the energy come from to sustain current after switch is flipped?

Answer: The magnetic field inside the inductor stores energy.

Recall power $P = \frac{dU}{dt} = iV \rightarrow i\mathcal{E}$, where U is energy.

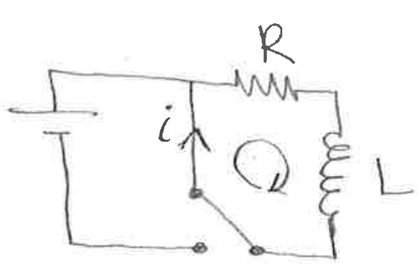
induced emf

$$\Rightarrow \text{Energy} = \int \frac{dU}{dt}\,dt = \int P\,dt = \int i\mathcal{E}\,dt$$

$$= \int i\left(L\frac{di}{dt}\right)dt = L\int i\,di = \boxed{\frac{1}{2}Li^2}$$

Stored energy

How does current depend on time?



Current continues flowing clockwise immediately after switch is flipped.

Kirchhoff Loop: $-Ri - L\frac{di}{dt} = 0$

Note that again the sign on $L\frac{di}{dt}$ is negative...

the clockwise loop is in the direction of current (that's all that matters)

Same method to solve :

$$-Ri = L\frac{di}{dt}$$

$$\Rightarrow -\frac{R}{L}dt = \frac{di}{i}$$

$$\Rightarrow \int_0^t -\frac{R}{L}dt = \int_{i(0)}^{i(t)} \frac{di}{i}$$

For convenience reset watch to be $t=0$ at new time of switch flip.

$$\Rightarrow -\frac{R}{L}t = \ln i \Big|_{i(0)}^{i(t)}$$

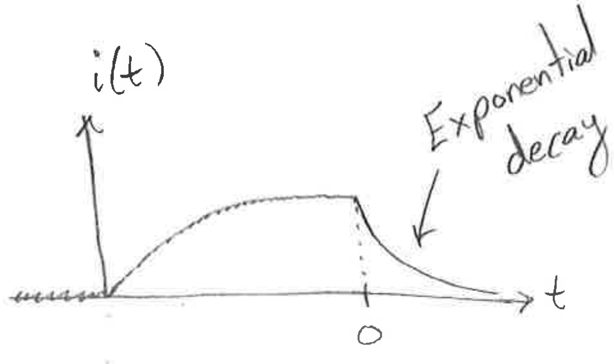
Here $i(0) = \frac{V}{R}$ since we left the original switch at 2 for a long time

$$\Rightarrow -\frac{R}{L}t = \ln i(t) - \ln\frac{V}{R}$$

$$\Rightarrow -\frac{R}{L}t = \ln\left(\frac{Ri}{V}\right)$$

$$\Rightarrow e^{-\frac{R}{L}t} = \frac{Ri}{V}$$

$$\Rightarrow \boxed{i = \frac{V}{R}e^{-\frac{R}{L}t}}$$



⊛ Note that for both differential equations, we needed an "initial condition" (that is, the value of the current before the switch was flipped)

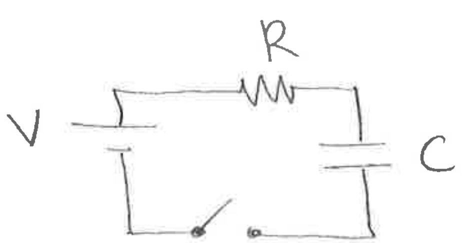
$i(0) = 0$ in the first case

$i(0) = \frac{V}{R}$ in the second case

This "initial condition" is always needed to solve a differential equation.

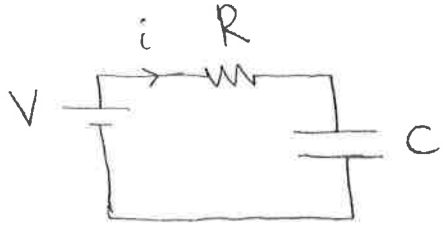
12.3) RC circuits

Let's now revisit RC circuits and study their time dependence (before we always assumed circuit had been connected for a long time).



Suppose at $t=0$ switch is closed.

⊛ Again, just use Kirchhoff's Loop Rule for an arbitrary time.



Now, both $Q(t)$ on the capacitor and $i(t)$ depend on time

(*) We expect $Q(t)$ to gradually increase with time and $i(t)$ to gradually decrease with time.

Kirchhoff : $V - iR - \frac{Q}{C} = 0$

But how are i and Q related?

Q is amount of charge that has left the battery:

$$\boxed{i(t) = \frac{dQ}{dt}}$$

← "Charging" capacitor. See 12.14 for "discharging" equation.

$$\Rightarrow V - R\frac{dQ}{dt} - \frac{Q}{C} = 0$$

(same basic type of differential equation seen in LR circuits)

$$\Rightarrow V - \frac{Q}{C} = R\frac{dQ}{dt}$$

$$\Rightarrow \frac{1}{R}\,dt = \frac{dQ}{V - \frac{1}{C}Q}$$

$$\Rightarrow \frac{1}{R}\int_0^t dt = \int_{Q(0)}^{Q(t)} \frac{dQ}{V-\frac{1}{C}Q}$$

$$\Rightarrow \frac{t}{R} = -C\ln\left(V-\frac{Q}{C}\right)\Bigg|_{Q(0)}^{Q(t)}$$

initially uncharged so $Q(0)=0$.

$$\Rightarrow -\frac{t}{RC} = \ln\left(V-\frac{Q}{C}\right) - \ln(V)$$

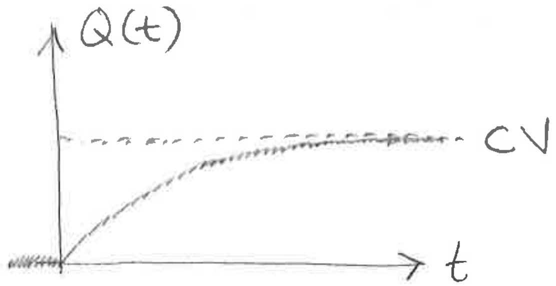
$$\Rightarrow -\frac{t}{RC} = \ln\left(1-\frac{Q}{CV}\right)$$

$$\Rightarrow e^{-t/RC} = 1-\frac{Q}{CV}$$

$$\Rightarrow Q(t) = CV\left(1-e^{-t/RC}\right)$$

(*) Makes sence : at $t=0$, $Q(0) = CV(1-1) = 0$

at $t\to\infty$, $Q(\infty) = CV(1-0) = CV$

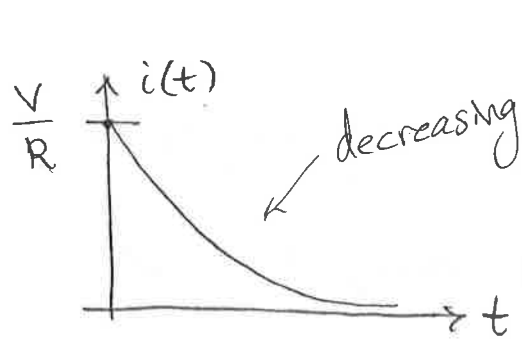


Current found by differentiating :

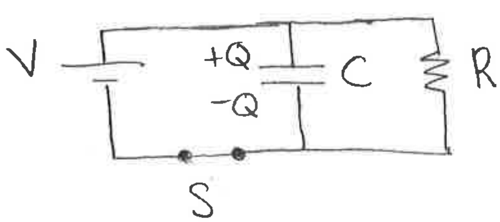
$$i(t) = \frac{dQ}{dt} = \frac{d}{dt}\left[CV\left(1 - e^{-t/RC}\right)\right]$$

$$i(t) = -CV\left(\frac{-1}{RC}e^{-t/RC}\right)$$

$$i(t) = \frac{V}{R}e^{-t/RC}$$



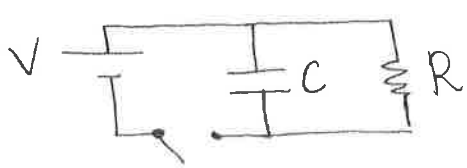
What about a discharging a capacitor?



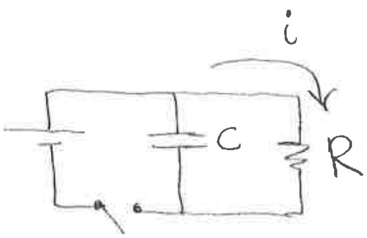
$$V - \frac{Q}{C} = 0$$

$$\Rightarrow Q = VC$$

Now open switch S :



Capacitor will discharge through resistor R.



Kichhoff loop: right loop clockwise

$$\frac{Q(t)}{c} - Ri(t) = 0$$

(*) Crucial point: For a discharging capacitor the current $i(t)$ (which is positive) must be

$$\boxed{i(t) = -\frac{dQ}{dt}}$$ (since Q is decreasing)

↑ Discharging

$$\Rightarrow \quad \frac{Q}{C} + R\frac{dQ}{dt} = 0$$

$$\Rightarrow -\frac{Q}{C} = R\frac{dQ}{dt}$$

$$\Rightarrow -\frac{1}{RC}t = \frac{dQ}{Q}$$

$$\Rightarrow \quad -\frac{1}{RC}\int_0^t dt = \int_{Q(0)}^{Q(t)} \frac{dQ}{Q}$$

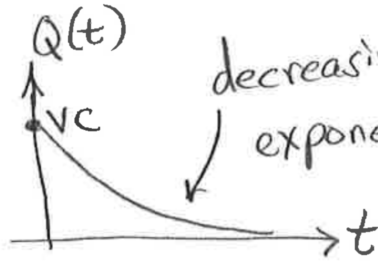
$$\Rightarrow \quad -\frac{1}{RC}t = \ln Q \Big|_{Q(0)}^{Q(t)} \leftarrow Q(0) = VC$$

$$\Rightarrow \quad -\frac{1}{RC}t = \ln Q - \ln(VC)$$

$$\Rightarrow \quad -\frac{t}{RC} = \ln \frac{Q}{VC}$$

$$\Rightarrow \quad e^{-t/RC} = \frac{Q}{VC}$$

$$\Rightarrow \quad Q(t) = VC\,e^{-t/RC}$$



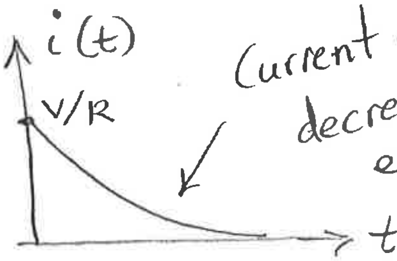
decreasing exponential

Current:

$$\Rightarrow \quad i(t) = -\frac{dQ}{dt}$$

$$\Rightarrow \quad i(t) = -VC\left(\frac{-1}{RC}\right)e^{-t/RC}$$

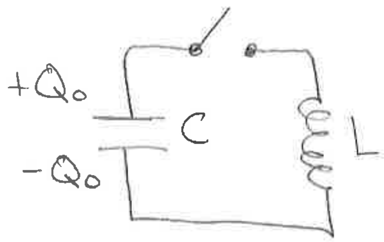
$$\Rightarrow \quad i(t) = \frac{V}{R}e^{-t/RC}$$



Current also decreases exponentially.

12.4) LC Circuits

Suppose we charge up a capacitor and then connect it in series with an inductor.



At time $t=0$ we close the switch.

Kirchhoff's Loop Rule:

Since charge flows clockwise, let's take our loop in that direction

$$+V_c - L\frac{di}{dt} = 0$$

$$\frac{Q}{C} = L\frac{di}{dt}$$

(Remember always $-L\frac{di}{dt}$ when loop is in direction of current)

Question: How is i related to the charge Q on the capacitor?

Not $i = \frac{dQ}{dt}$

Since $Q(t)$ is the charge remaining on the capacitor at time t, then $q(t) = Q_0 - Q(t)$ must be the charge that has left.

$$\Rightarrow \quad i = \frac{dq}{dt} = -\frac{dQ}{dt}$$ (correct relationship between Q and i)

$$\Rightarrow \quad \frac{Q}{C} = -L\frac{d^2Q}{dt^2}$$

$$\frac{d^2Q}{dt^2} = -\frac{1}{LC}Q$$

$$\Rightarrow \quad Q(t) = \underbrace{A\sin\left(\frac{1}{\sqrt{LC}}t\right) + B\cos\left(\frac{1}{\sqrt{LC}}t\right)}_{\text{General solution}}$$

Initial Conditions

(1) At $t = 0$, $Q(0) = Q_0$

$$\Rightarrow \quad Q_0 = 0 + B(1)$$

$$\Rightarrow \quad \boxed{B = Q_0}$$

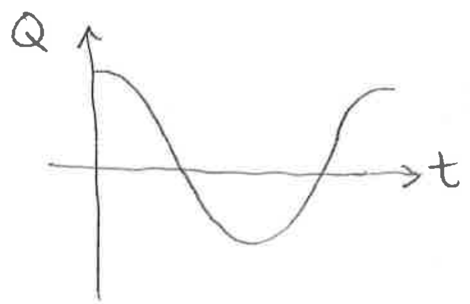
(2) At $t=0$, $i = -\frac{dQ}{dt} = 0$

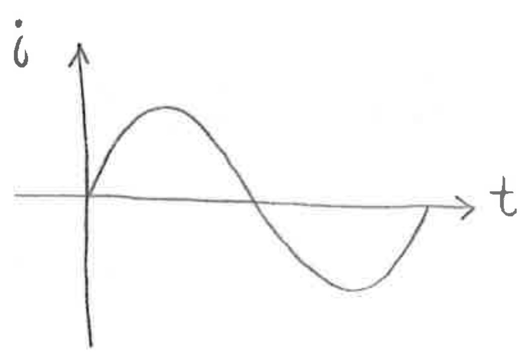
$$\Rightarrow \frac{A}{\sqrt{LC}} \cos\left(\frac{1}{\sqrt{LC}}(0)\right) - \frac{B}{\sqrt{LC}} \sin\left(\frac{1}{\sqrt{LC}}(0)\right) = 0$$

$$\Rightarrow \boxed{A = 0}$$

$$\Rightarrow \boxed{Q(t) = Q_0 \cos\left(\frac{1}{\sqrt{LC}} t\right)}$$

⊛ Current won't go to zero in this idealized case, instead it just oscillates back and forth

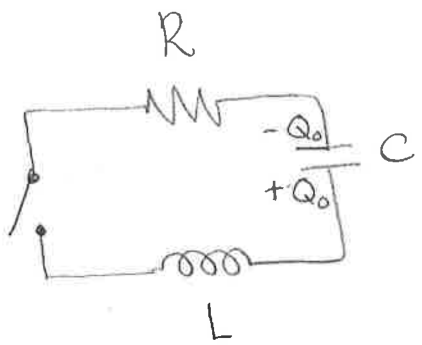




What is the maximal current?

$$i = -\frac{dQ}{dt} = -\frac{Q_0}{\sqrt{LC}} \sin\left(\frac{1}{\sqrt{LC}} t\right) \longrightarrow \boxed{\frac{Q_0}{\sqrt{LC}}}$$

LRC circuits



The capacitor starts out charged ($+Q_0$) and at time $t=0$ the switch is closed

Kirchhoff's Loop Rule

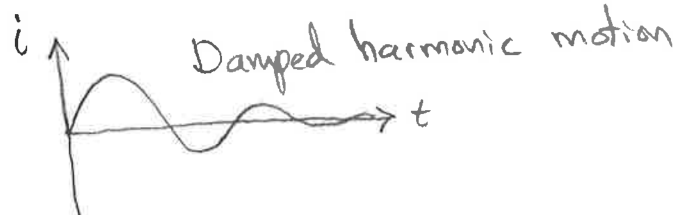
Current will flow clockwise, so take loop path in that direction

$$\frac{Q}{C} - L\frac{di}{dt} - iR = 0$$

$$\frac{Q}{C} + L\frac{d^2Q}{dt^2} + \frac{dQ}{dt}R = 0$$

(Rather complicated differential equation.)

If R is relatively small $\Rightarrow$



If R is relatively large $\Rightarrow$