PHYS 206 Lecture 11

11.1) Oscillating systems

⊛ Many objects in the real world undergo oscillatory motion.

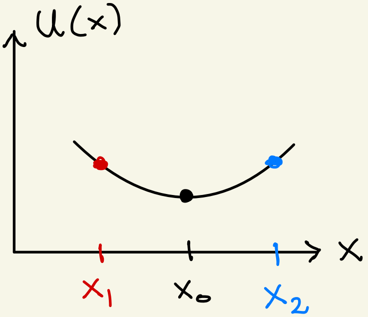
→ Pull on a tree branch, and it will return to its original position after undergoing a few oscillations.

→ Drop a rock in a lake and the water waves will oscillate in time.

→ Springs

→ Pendulum

⊛ Oscillating systems are common because systems are often at the minimum of potential energy and once disturbed they want to return to that minimum.



$$F(x_1) = -\frac{dU}{dx} > 0$$

$\Rightarrow F(x_1)$ is to the right

$$F(x_2) = -\frac{dU}{dx} < 0$$

$\Rightarrow F(x_2)$ is to the left

$$F(x_0) = -\frac{dU}{dx} = 0 \quad \text{(equilibrium)}$$

11.2) Differential equation for oscillations

(*) Taylor series expansion of the potential energy function about its minimum value:

$$U(x) = U(x_0) + \underbrace{U'(x_0)}_{=0 \text{ since } x_0 \text{ is at minimum}}(x-x_0) + \frac{1}{2}U''(x_0)(x-x_0)^2 + \cdots$$

$$= U(x_0) + \frac{1}{2}U''(x_0)(x-x_0)^2 + \cdots$$

$$\Rightarrow F(x) = -\frac{dU}{dx} = -U''(x_0)(x-x_0) + \cdots$$

Normally convenient to define origin to be at x_0 (like we did with springs):

$$F(x) = -kx + \cdots \quad (\text{where } k = U''(x_0) > 0)$$

⊛ Fundamental equation for oscillations:

$$\boxed{F = -Kx}$$

This is a

(1) linear (proportional to x)

(2) restoring (opposite to displacement)

type of force.

⊛ From Newton's 2nd Law:

$$\vec{F} = m\vec{a}$$

$$-Kx = m\frac{d^2x}{dt^2} \quad \text{(differential equation)}$$

position depends on time: $x(t)$

2nd derivative of $x(t)$

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} = -\frac{K}{m}\,x(t)}$$

⊛ What function $x(t)$ has a second derivative proportional to − itself??

Answer: Some combination of sin and cos.

Let $x(t) = A\sin(\omega t) + B\cos(\omega t)$.

Then $\frac{dx}{dt} = A\omega\cos(\omega t) - B\omega\sin(\omega t)$

and $\frac{d^2x}{dt^2} = -\omega^2 A\sin(\omega t) - \omega^2 B\cos(\omega t)$

$$= -\omega^2\left[A\sin(\omega t) + B\cos(\omega t)\right]$$

$$= -\omega^2\left[x(t)\right]$$

In order for Newton's 2nd Law to be satisfied

$$\omega = \sqrt{\frac{K}{m}}.$$

General solution:

$$\boxed{x(t) = A\sin\left(\sqrt{\frac{K}{m}}\,t\right) + B\cos\left(\sqrt{\frac{K}{m}}\,t\right)}$$

A and B are undetermined constants that are fixed by the initial conditions of the system.

11.3) Applications

⊛ Simplest application is to springs, for which $\vec{F}(x) = -Kx\,\hat{i}_x$.

Example: A block of mass m is connected to a spring with spring constant K that is compressed a distance d and released from rest. What is the resulting position of the block as a function of time?

Solution:

$$F = -Kx = ma = m\frac{d^2x}{dt^2}$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} = \left(-\frac{K}{m}\right)x$$

$\Rightarrow$ General solution is

$$x(t) = A\sin\left(\sqrt{\tfrac{K}{m}}\,t\right) + B\cos\left(\sqrt{\tfrac{K}{m}}\,t\right)$$

⊛ Note that the initial conditions are

(i) "compressed a distance d" $\Rightarrow x(0) = -d$

(ii) "released from rest" $\Rightarrow v(0) = 0$

Plugging in we find

$$x(0) = -d$$

$$A\sin\left(\sqrt{\frac{K}{m}}\,0\right) + B\cos\left(\sqrt{\frac{K}{m}}\,0\right) = -d$$

$$B = -d$$

Since $x(t) = A\sin\left(\sqrt{\frac{K}{m}}\,t\right) + B\cos\left(\sqrt{\frac{K}{m}}\,t\right)$,

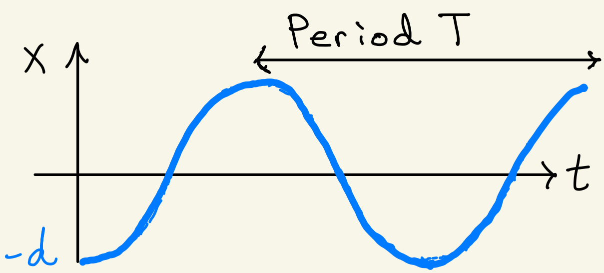
$$v(t) = A\sqrt{\frac{K}{m}}\cos\left(\sqrt{\frac{K}{m}}\,t\right) - B\sqrt{\frac{K}{m}}\sin\left(\sqrt{\frac{K}{m}}\,t\right)$$

Therefore, $v(0) = 0$

$$\Rightarrow \quad A\sqrt{\frac{K}{m}}\cos\left(\sqrt{\frac{K}{m}}\,0\right) - B\sqrt{\frac{K}{m}}\sin\left(\sqrt{\frac{K}{m}}\,0\right) = 0$$

$$\Rightarrow \quad A\sqrt{\frac{K}{m}} = 0 \Rightarrow A = 0$$

$$\Rightarrow \quad x(t) = -d\cos\left(\sqrt{\frac{K}{m}}\,t\right)$$



Period given by

$$\sqrt{\frac{K}{m}}\,T = 2\pi$$

$$\boxed{\Rightarrow T = \frac{2\pi}{\omega}}$$

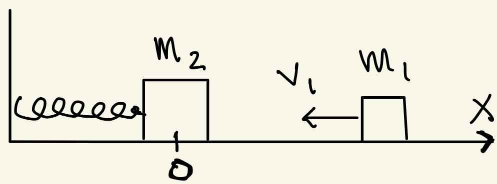
From $x(t)$ we can of course calculate also $v(t)$ and $a(t)$:

$$v(t) = d\sqrt{\frac{K}{m}}\,\sin\left(\sqrt{\frac{K}{m}}\,t\right), \text{ etc.}$$

(*) There are lots of different variations depending on the initial conditions.

<u>Example</u>: Block m_2 at rest on spring. An incoming block m_1 with speed v_1 collides and sticks to m_2. What is the motion after collision? Neglect any friction.

<u>Solution</u>:



Momentum conservation gives

$$-m_1 v_1 = (m_1 + m_2)\,v$$

$$\Rightarrow\ v(0) = -\frac{m_1}{m_1+m_2} v_0 \quad (\text{initial condition } 1)$$

$$x(0) = 0 \quad (\text{initial condition } 2)$$

Now Newton's 2nd Law gives

$$\vec{F} = m\vec{a} \Rightarrow -Kx = (m_1+m_2)\frac{d^2x}{dt^2}$$

$$\Rightarrow \frac{d^2x}{dt^2} = \left(-\frac{K}{m_1+m_2}\right)x$$

Same general solution as before

$$x(t) = A\sin\left(\sqrt{\frac{K}{m}}\,t\right) + B\cos\left(\sqrt{\frac{K}{m}}\,t\right) \qquad m \equiv m_1+m_2$$

But now $x(0) = 0 \Rightarrow B = 0$

And $v(0) = \dfrac{-m_1}{m_1+m_2}v_1$

$$\Rightarrow A\sqrt{\frac{K}{m}} = \frac{-m_1}{m_1+m_2}v_1$$

$$\Rightarrow A = -\sqrt{\frac{m_1+m_2}{K}}\left(\frac{m_1}{m_1+m_2}\right)v_1$$

(*) The spring and mass properties (K, m) determine uniquely the period and frequency.

(*) The <u>amplitude</u> of motion <u>does</u> <u>not</u> <u>affect</u> the period or frequency.

* **Angular frequency (ω)**: Number of radians per second.

For $x(t) = A\sin\left(\sqrt{\frac{K}{m}}\,t\right) + B\cos\left(\sqrt{\frac{K}{m}}\,t\right)$, the phase in radians after 1 second is

$$\boxed{\omega = \sqrt{K/m}}$$

Related quantity: "frequency" (f) is the number of full cycles (2π) per second.

$$\boxed{f = \frac{\omega}{2\pi}}$$

Phase after 1 second. If $\omega = 4\pi$, then there will be 2 cycles/sec.

Period (T): time to complete one full cycle (2π) of motion.

For $x(t) = A\sin\left(\sqrt{\frac{K}{m}}\,t\right) + B\cos\left(\sqrt{\frac{K}{m}}\,t\right)$

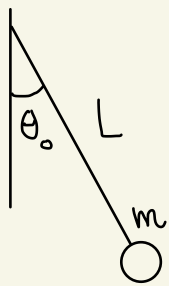
$$\sqrt{\frac{K}{m}}\,T = 2\pi$$

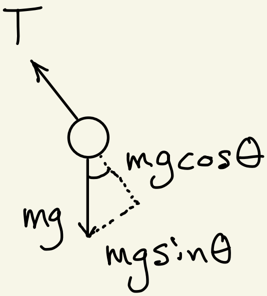
$$\Rightarrow T = \sqrt{\frac{m}{K}}\,2\pi$$

$$\boxed{T = \frac{2\pi}{\omega}}$$

<u>Example</u>: Consider a mass m that hangs from a nearly massless rope of length L. If the mass is pulled through a small angle θ_o and released from rest, what will be the resulting motion $\theta(t)$?

<u>Solution</u>:





* Since mass does not accelerate radially,

$$T = mg\cos\theta$$

* The tangential acceleration is

$$\left(\alpha L + 2\omega \cancelto{0}{\frac{dr}{dt}}\right)\hat{i}_\theta = \alpha L \hat{i}_\theta$$

From $F_\theta = ma_\theta$ we find

$$-\cancel{m}g\sin\theta = \cancel{m}L\frac{d^2\theta}{dt}$$

$$\Rightarrow \quad \frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

In the small angle approximation

$$\sin\theta \approx \theta .$$

Therefore we find the differential equation describing the motion of a pendulum:

$$\boxed{\frac{d^2\theta}{dt^2} = \left(-\frac{g}{L}\right)\theta}$$

Note that this is exactly the same form as

$$\frac{d^2x}{dt^2} = \left(-\frac{K}{m}\right)x$$

⊛ General solution:

$$\theta(t) = A\sin\left(\sqrt{\frac{g}{L}}\,t\right) + B\cos\left(\sqrt{\frac{g}{L}}\,t\right)$$

Initial condition 1) $\theta(0) = \theta_0$

Initial condition 2) $\omega(0) = 0$

$$\Rightarrow \theta(0) = B\cos(0) = \underline{B = \theta_0}$$

$$\Rightarrow \omega(0) = A\sqrt{\frac{g}{L}}\cos(0) = A\sqrt{\frac{g}{L}} = 0 \Rightarrow \underline{A = 0}$$

The complete motion at any time is therefore

$$\theta(t) = \theta_0 \cos\left(\sqrt{\frac{g}{L}}\, t\right)$$

→ angular frequency $\omega = \sqrt{\frac{g}{L}}$

→ period $T = 2\pi\sqrt{\frac{L}{g}}$

* The period is larger for large L or small g.

What is the consequence for how fast you can walk on Earth?

Answer → How fast you walk is related to the natural period of your swinging leg. Therefore, you will walk <u>slower</u> on the Moon since the period of pendulum motion is larger.

$$g_{moon} \approx \frac{1}{6}\, g_{Earth}$$