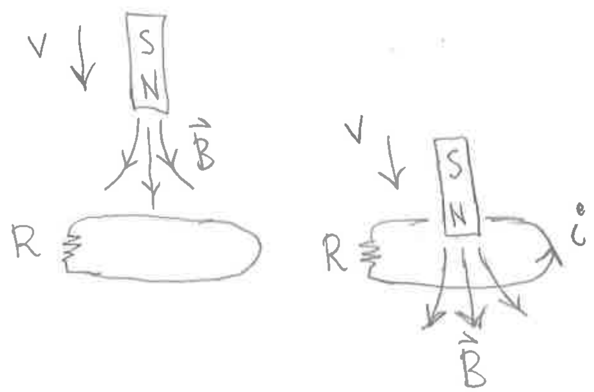
PHYS 207 Electricity and Magnetism

11. Electromagnetic Induction

11.1 Faraday's Law
11.2 Induced Electric Fields
11.3 Motional EMF

11.1) Faraday's Law

* Experimentally we find that when we move a magnet in the vicinity of a closed conducting wire loop, something interesting happens:



There is a current induced in the wire even though no battery is attached!

* Precise experiments reveal the electromotive force (emf)

$$\boxed{\mathcal{E} = -\frac{d\phi_B}{dt}}$$ (Faraday's Law)

"Induced emf" (voltage)

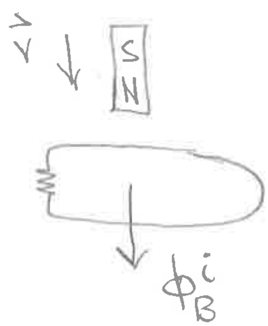
rate of change of magnetic flux ϕ_B.

Let's break down this equation

- **Induced emf ($\mathcal{E}$)**: just like batteries have an emf that pushes charges around a circuit, an induced emf is created by a changing magnetic flux ϕ_B

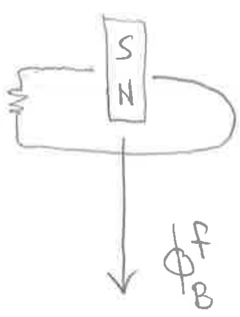
- The direction of the induced emf and the resulting current can be found by religiously following the steps:

  (a) Draw the initial flux "vector" through the loop (flux isn't actually a vector, but it does have a directional sign, so let's just imagine a vector)



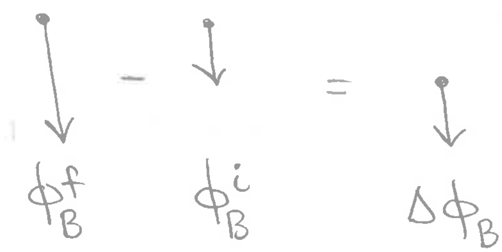
$$\phi_B = \int \vec{B} \cdot d\vec{A}$$

  (b) Draw the final flux direction and magnitude a short time later:

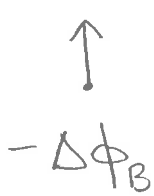


(larger than ϕ_B^i since $\vec{B}$ field is larger and area stays the same)

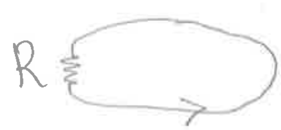
(c) Find the direction of $\Delta\phi_B = \phi_B^f - \phi_B^i$



(d) Determine the direction of $-\Delta\phi_B$ $\left(\text{because } \mathcal{E} = -\frac{d\phi_B}{dt}\right)$



(e) The induced current is such that it produces a magnetic field in the direction $-\Delta\phi_B$



i travels counterclockwise viewed from above

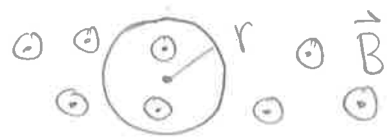
(*) Magnitude of current i:

$$\mathcal{E} = \frac{d\phi_B}{dt} = iR$$

$$i = \frac{1}{R}\frac{d\phi_B}{dt}$$

Example : (Exam 3, 2011, Problem 2)

A circular loop of wire with restivity ρ has radius r and cross sectional area A. A uniform magnetic field $\vec{B} = B_o \sin(\beta t)\, \hat{i}_x$ is perpendicular to the plane of the loop. Find the induced current.



Solution: First compute resistance :

$$R = \rho \frac{L}{A} = \rho \frac{2\pi r}{A}$$

Next calculate $\mathcal{E}_{ind} = \dfrac{d\phi_B}{dt}$. Note that

$$\phi_B = \int \vec{B}\cdot d\vec{A} = \int B_o \sin(\beta t)\, dA$$

$$= B_o \sin(\beta t)\, \pi r^2$$

$$\Rightarrow \frac{d\phi_B}{dt} = \pi r^2 B_o \beta \cos(\beta t)$$

$$\Rightarrow \mathcal{E} = \pi r^2 B_o \beta \cos(\beta t) = iR = i\left(\rho \frac{2\pi r}{A}\right)$$

$$\Rightarrow \boxed{i = \frac{A B_o \beta r \cos(\beta t)}{2\rho}}$$

11.2) Induced Electric Fields

(*) Faraday's Law is completely general and applies to cases for which

(a) $\vec{B}$ field changes but wire loop is fixed in place

(b) $\vec{B}$ field is constant but wire loop changes

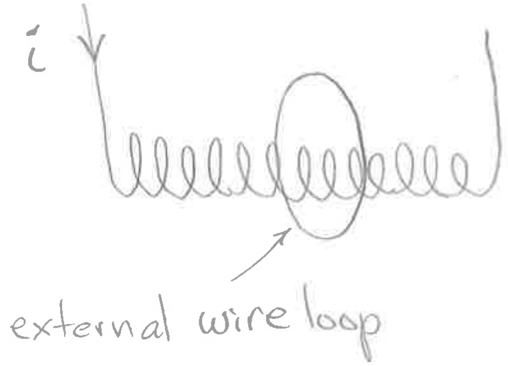
Both (a) and (b) result in a changing magnetic flux

$$\phi_B = \int \vec{B}\cdot d\vec{A}$$

(*) The microscopic origin of the EMF produced in case (a) is the following:

$$\boxed{\text{Time-dependent } \vec{B} \text{ fields induce } \vec{E} \text{ fields}}$$

Example: Consider a tightly-wrapped solenoid carrying a current $i(t)$ increasing with time. Place a wire around the solenoid.

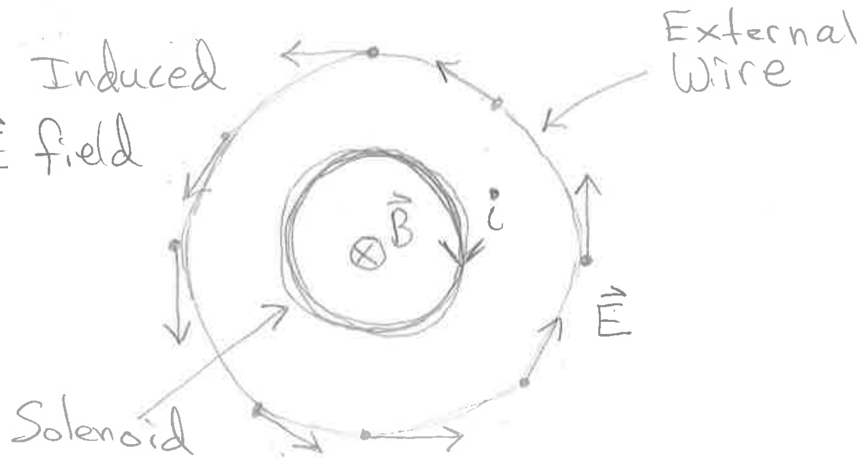


(*) $\vec{B}$ fields outside of solenoids cannot affect the stationary charges in the external loop

However, by Faraday's Law $\mathcal{E} = -\frac{d\phi_B}{dt}$

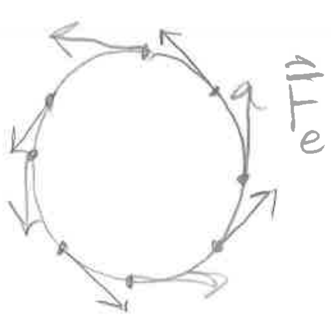
there must be a current induced in the wire since $\vec{B}$ is increasing

Apparently an electric field is created around the solenoid and by symmetry it must circulate



⊛ These $\vec{E}$ fields have a peculiar characteristic:

$\boxed{\vec{F}_e \text{ is } \underline{\text{not}} \text{ conservative}}$



clearly $\oint \vec{F}_e \cdot d\vec{r} \neq 0$

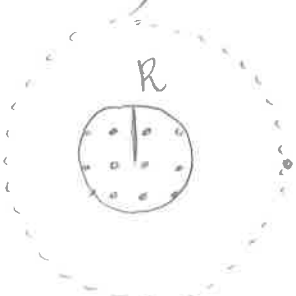
(*) This is why we say there is an induced EMF, rather than an induced potential difference ΔV (electric potential only defined for conservative force)

(*) But we can still define

$$\mathcal{E}_{ind} = \oint \vec{E} \cdot d\vec{r} \qquad \left(\text{compare } \Delta V = -\int \vec{E} \cdot d\vec{r}\right)$$

<u>Example</u>: Suppose there is a uniform magnetic field confined to a circular area with radius R. If $B(t) = B_0 \sin(\alpha t)$, what is the magnitude of the induced electric field a distance r away from the center?

(Solution) From Faraday's Law $\mathcal{E} = -\dfrac{d\phi_B}{dt}$



$$\Rightarrow \oint \vec{E} \cdot d\vec{r} = -\frac{d\phi_B}{dt}$$

$$E(2\pi r) = -\frac{d}{dt}\left[B_0 \sin(\alpha t)\,\pi R^2\right]$$

$$E(2\pi r) = -B_0 \alpha \cos(\alpha t)\,\pi R^2$$

$$E = -\frac{1}{2\pi r}\left[B_0 \alpha \cos(\alpha t)\, \pi R^2\right]$$

$$E = -\frac{1}{2r} B_0 \alpha R^2 \cos(\alpha t)$$

Direction of current at $t = 0$?

ϕ_B^i	ϕ_B^f	$\Delta \phi_B$	$-\Delta \phi_B$
0	$\odot$	$\odot$	$\otimes$

$\vec{E}$ must circulate clockwise to produce a $\vec{B}$ field in the $\otimes$ direction.

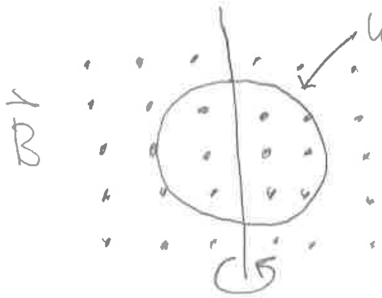
11.3) Motional EMF

An induced current can be produced when the shape of a conductor in a $\vec{B}$ field changes or if the conductor moves

⊛ This is called "motional EMF"

⊛ Microscopically it results from the magnetic forces acting on the mobile charges in the conductor

Example 1

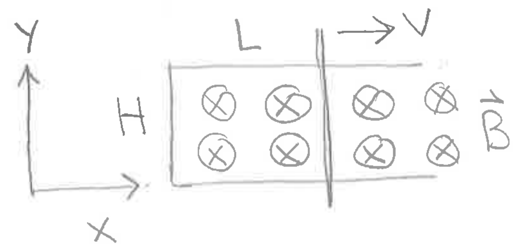


If the loop rotates, the magnetic flux will oscillate $\Rightarrow \mathcal{E} = -\dfrac{d\phi_B}{dt}$

Example 2:

Suppose there is a set of metal wires all touching each other and in a uniform background magnetic field $\vec{B} = -B_o \hat{\imath}_z$. If the right wire moves with velocity $\vec{v} = v\,\hat{\imath}_x$, what is the induced EMF?

(Solution)



Since the magnetic flux is increasing, there will be an induced EMF given by

$$\mathcal{E} = -\frac{d\phi_B}{dt}$$

$$\phi_B = \int \vec{B} \cdot d\vec{A}$$

$$= B_o \int dA$$

$$= B_o H \underbrace{L(t)}_{\text{length depends on time}}$$

$$\Rightarrow \mathcal{E} = -\frac{d}{dt}\left[B_0 H L(t)\right]$$

$$\mathcal{E} = -B_0 H \frac{dL}{dt}$$

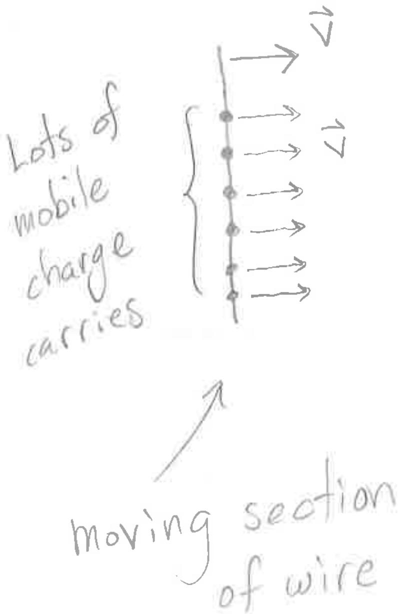
$$\boxed{\mathcal{E} = -B_0 H v}$$

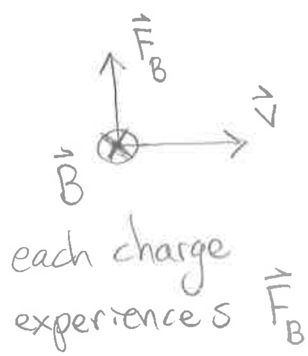
Direction of induced current

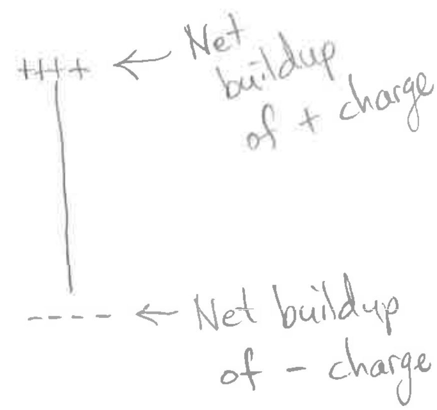
Φ_B^i	Φ_B^f	$\Delta\Phi_B$	$-\Delta\Phi_B$
$\otimes$	$\bigotimes$	$\otimes$	$\odot$

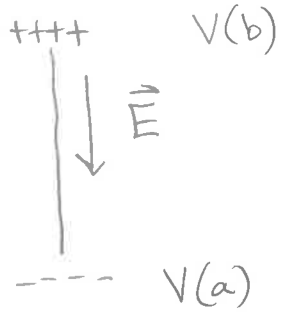
Counterclockwise current will produce $\vec{B}$ field in $\odot$ direction.

What is the microscopic origin of this induced emf and current?









This buildup of charge produces an $\vec{E}$ field until

$$\boxed{\vec{F}_B = -\vec{F}_E}$$

At which point the forces balance.

Since $\vec{F}_B = q\vec{v} \times \vec{B} = q v B_0 \hat{i}_y$ and $\vec{F}_E = q\vec{E} = qE(-\hat{i}_y)$,

$$vB_0 = E$$

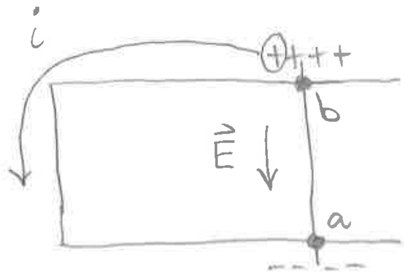
$$\Rightarrow V(b) - V(a) = -\int_a^b \vec{E} \cdot d\vec{r} = -\int_a^b -E\,dr$$

$$= \int_a^b vB_0\,dl = \boxed{vB_0 H}$$

(*) Note that this exactly what we obtained for $\mathcal{E}_{ind}$ using Faraday's Law!

(*) Note: using Faraday's Law was simpler!

What about the direction of current flow from the microscopic picture?

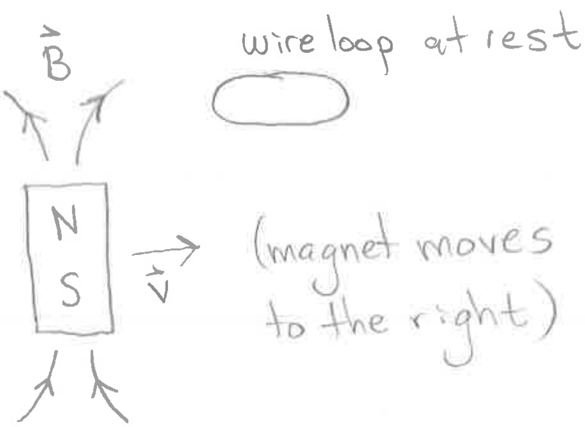


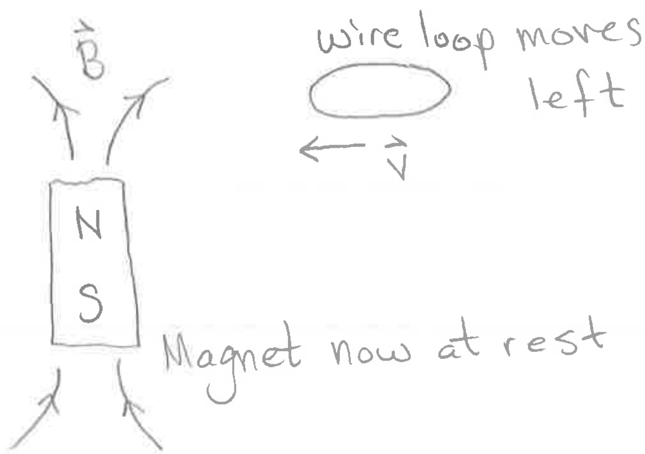
$V(b) > V(a)$

Positive charges will flow around <u>counterclockwise</u> to lower their potential energy (same as from Lenz's Law)

Once this happens the $\vec{E}$ field between b and a will be reduced, causing $\vec{F}_B > \vec{F}_E$ and an extra ⊕ charge to flow from a to b.

<u>Extra:</u> Here is something to blow your mind. Consider these two scenarios where the <u>relative</u> motion between magnet and wire are identical:





Current will flow in stationary wire loop due to an induced <u>electric</u> <u>field</u> around magnet.

Current will flow in moving wire loop due to the <u>magnetic</u> <u>forces</u> on mobile charges.