10.) Sources of magnetic fields

10.1) $\vec{B}$ fields due to moving point charges

10.2) $\vec{B}$ fields due to currents

10.3) Ampere's Law

10.1) Magnetic fields caused by moving charges

⊛ Key Point: All magnetic fields result from charges in motion.

⊛ Fundamental Law

unit vector $\left(\hat{r} = \frac{1}{r}\vec{r}\right)$

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{q\,\vec{v} \times \hat{r}}{r^2}$$

This is actually a bit tricky to interpret, so let's break it down:

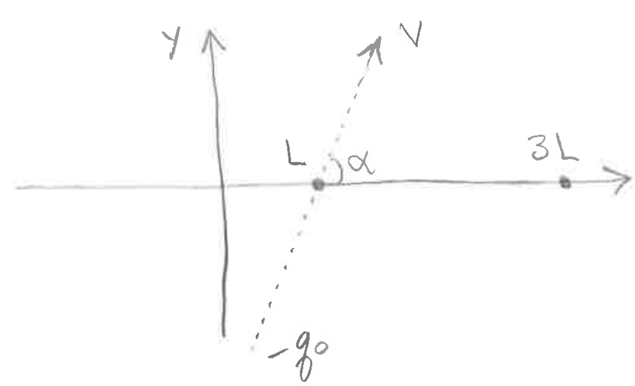
(a) The constant μ_0 is called the "permeability of free space", analogous to ε_0 for electric fields

(b) q is just the charge of the particle in motion (sign matters!)

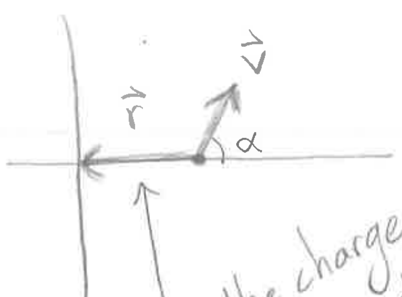
(c) $\vec{v}$ is the instantaneous velocity vector

(d) $\vec{r}$ is the vector from the location of the charge to the location where we want to measure $\vec{B}$.

Example: Consider the dotted trajectory of a charge $-q_0$ with speed v as shown. At the instant the charge crosses the x-axis, what is the instantaneous magnetic field at the origin? At the point $3L\hat{i}_x + 0\hat{i}_y$?



Solution: At the instant $-q_0$ crosses the x-axis, we can draw the $\vec{v}$ and $\vec{r}$ vectors as follows



from the charge and to the point where we compute $\vec{B}$.

Note that $\hat{r}$ points toward the origin.

$$\vec{v} \times \hat{r} = |\vec{v}||\hat{r}| \sin\theta \odot$$

$$= (v)(1)\sin(\pi - \alpha) \odot$$

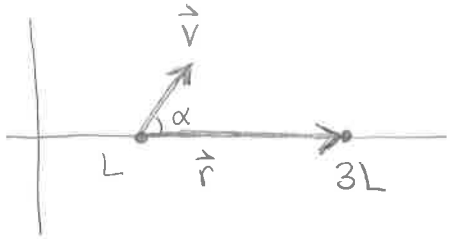
$$= v \sin\alpha \odot$$

out of page

$$\Rightarrow \vec{B} = \frac{\mu_0}{4\pi}(-q_0)\frac{v\sin\alpha}{L^2} \text{ out of page } = \boxed{\frac{\mu_0}{4\pi} q_0 \frac{v\sin\alpha}{L^2} \text{ into page}}$$

This is only the $\vec{B}$ field at the origin.

At the location $3L\hat{i}_x + 0\hat{i}_y$ we find



$$\vec{v} \times \hat{r} = |\vec{v}||\hat{r}|\sin\theta \otimes$$

$$= v(1)\sin\alpha \otimes$$

$$= v\sin\alpha \otimes \text{ into page}$$

$$\Rightarrow \vec{B} = \frac{\mu_0}{4\pi}(-q_0)\frac{v\sin\alpha}{(2L)^2} \otimes$$

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi}\frac{q_0 v\sin\alpha}{4L^2} \odot}$$

10.2) Magnetic fields due to currents

- In 1820 Hans Oersted discovered that a current-carrying wire caused a compass needle to move.

- From the point charge rule above, we can derive the total $\vec{B}$ field due to the many moving charges in a current:

$$d\vec{B} = \frac{\mu_0}{4\pi} dq \frac{\vec{v}\times\hat{r}}{r^2}$$

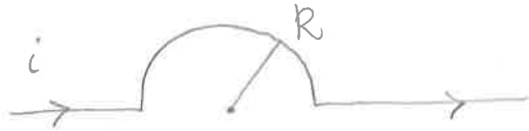
$$= \frac{\mu_0}{4\pi}(\rho_c dV)\frac{\vec{v}\times\hat{r}}{r^2} \quad (V = \text{volume},\ \vec{v} = \text{velocity})$$

$$= \frac{\mu_0}{4\pi}(\rho_c A\, d\ell)\frac{\vec{v}\times\hat{r}}{r^2}$$

$$= \frac{\mu_0}{4\pi}(\rho_c A v)\frac{d\vec{\ell}\times\hat{r}}{r^2} \quad (\vec{v} \text{ and } d\vec{\ell} \text{ are parallel})$$

$$\boxed{d\vec{B} = \frac{\mu_0}{4\pi}\, i\, \frac{d\vec{\ell}\times\hat{r}}{r^2}} \quad \text{Biot-Savart Law}$$

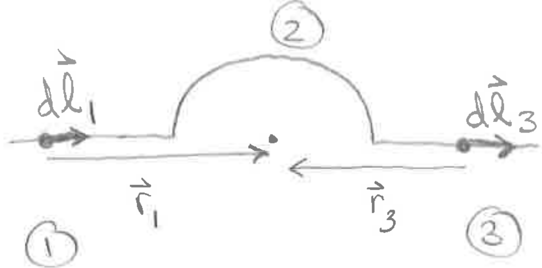
Example: Consider the current-carrying wire shown below, where the current i flows to the right and the circle radius is R. What is the $\vec{B}$ field at the center of the circle?



Solution:

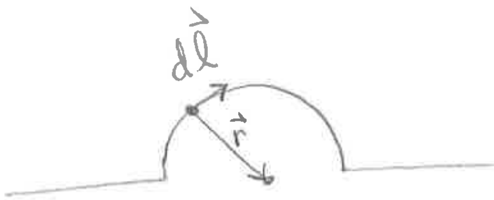
In principle we need to add up all contributions to the $\vec{B}$ field at the center due to the 3 separate sections of wire.

The straight sections turn out to be trivial because $d\vec{l}$ is parallel to $\vec{r}$



$$d\vec{l}_1 \times \vec{r} = 0$$
$$d\vec{l}_3 \times \vec{r} = 0$$

Only the semicircle part contributes to the $\vec{B}$ field at the center:



Note that tangent vector to circle is perpendicular to radius vector

$$d\vec{l}_2 \times \hat{r} = |d\vec{l}_2||\hat{r}| \sin\theta$$
$$= dl_2 (1)(1)$$
$$= dl_2$$

$$\Rightarrow \vec{B} = \int \frac{\mu_0}{4\pi} i \frac{d\vec{l} \times \hat{r}}{r^2}$$

$r = R$ (constant)

$$= \frac{\mu_0}{4\pi} \frac{i}{R^2} \underbrace{\int dl}_{\text{length of semicircle is } \pi R}$$

$$= \frac{\mu_0 i}{4\pi R^2}(\pi R) = \frac{\mu_0 i}{R}$$

⊛ For more complicated problems, use these 5 steps:

(1) Break up current into small pieces $i\,d\vec{l} = i\{d\vec{x}, d\vec{y}, Rd\theta\,\hat{i}_\theta\}$

(2) Determine integration region $\int dl$

(3) Compute $|d\vec{B}|$ from small current segment dl at an arbitrary point l within integration region:

$$|d\vec{B}| = \frac{\mu_0 i}{4\pi}\frac{|d\vec{l}\times\hat{r}|}{r^2} = \frac{\mu_0 i}{4\pi}\frac{|d\vec{l}|\sin\theta_{lr}}{r^2}$$

where θ_{lr} is the angle between $d\vec{l}$ and $\hat{r}$.

(4) Write θ_{lr} and/or r in terms of integration variable.

(5) Integrate and determine direction of $\vec{B}$ from right-hand-rule for $d\vec{l}\times\hat{r}$.

(5*) To integrate over θ for circular sections of wire, change $dl = Rd\theta$ and <u>always</u> choose increasing θ direction to be along current direction

Exam 2, 2015,

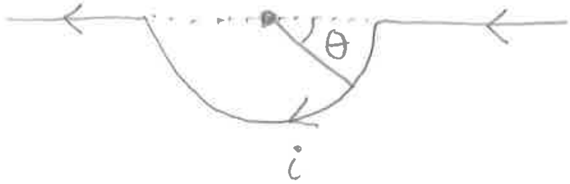
1. (25 points) A very thin wire lies in the x, y plane. It has the shape shown below consisting of two circular segments, centered at the origin, connected by segments along radii. There is a current i in the wire as shown. Find the magnetic field at the origin.

Law

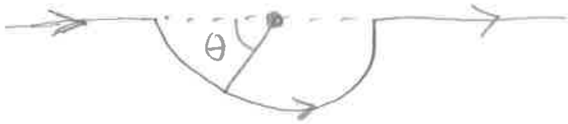
Application

Result

For instance,



$$\int_0^{\pi} d\theta$$



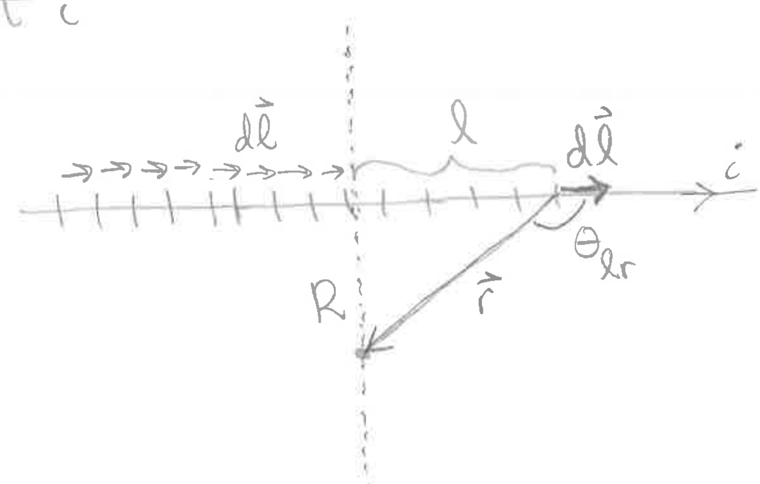
$$\int_0^{\pi} d\theta$$

$$\int_0^{\pi-\alpha} d\theta$$

Example:

Magnetic field a distance R from an infinitely long wire carrying current i

Solution:

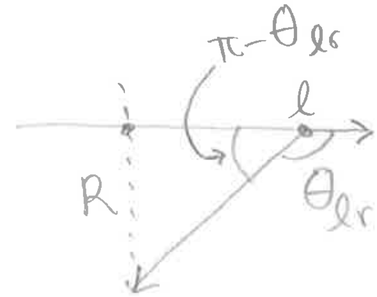
① 

② $\int_{-\infty}^{\infty} dl$

③ $|d\vec{B}|$ due to small current segment $i\,d\vec{\ell}$ at the arbitrary point ℓ:

$$dB = \frac{\mu_0}{4\pi}\, i\, \frac{d\vec{\ell} \times \hat{r}}{r^2}$$

④
$$= \frac{\mu_0 i}{4\pi} \frac{d\ell \sin\theta_{\ell r}}{\sqrt{R^2+\ell^2}}$$

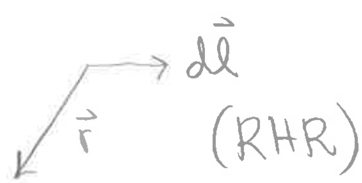


$$= \frac{\mu_0 i}{4\pi} \frac{d\ell \sin(\pi-\theta_{\ell r})}{\sqrt{R^2+\ell^2}}$$

$$= \frac{\mu_0 i}{4\pi} \frac{d\ell}{\sqrt{R^2+\ell^2}} \frac{R}{\sqrt{R+\ell^2}}$$

⑤
$$B = \frac{\mu_0 i R}{4\pi} \int_{-\infty}^{\infty} \frac{d\ell}{(R^2+\ell^2)^{3/2}} = \frac{\mu_0 i R}{4\pi} \left[\frac{2}{R^2} \right]$$

$$\boxed{\vec{B} = \frac{\mu_0 i}{2\pi R} \text{ (into page)}}$$

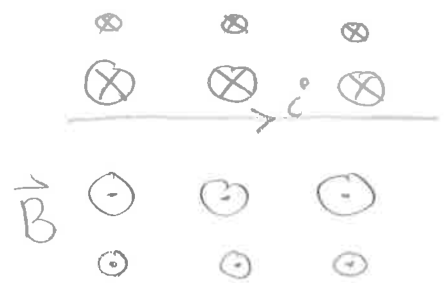


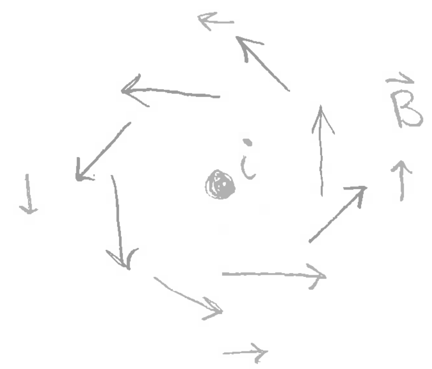
10.3) Ampere's Law

The above result for an infinitely long current carrying wire was fairly involved.

There is a much easier way to obtain same result.

(*) Key Point: Magnetic fields "circulate" around current-carrying wires



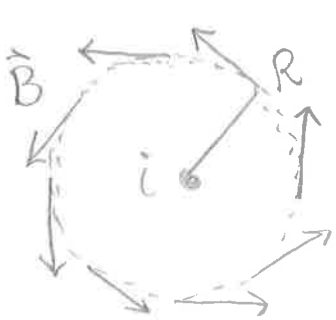


i out of page

(*) RHR: Point thumb in direction of current, then fingers curl in direction of $\vec{B}$ field circulation.

(*) The curling of a vector field can be measured with a closed line integral: $\oint \vec{B} \cdot d\vec{l}$.

Example: For the case above with the current flowing out of page, how does $\oint \vec{B} \cdot d\vec{l}$ vary with the choice of the radius R of the circulating path?



$B = \frac{\mu_0 i}{2\pi R}$ (proved previously from Biot-Savart)

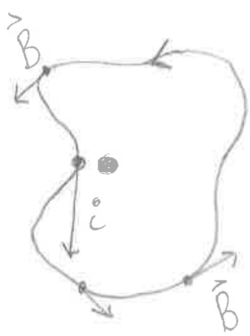
$$\oint \underbrace{\vec{B}\cdot d\vec{l}} = \int_0^{2\pi R} B\,dl = \int_0^{2\pi R} \frac{\mu_0 i}{2\pi R}\,dl = \frac{\mu_0 i}{2\pi R}(2\pi R)$$

$\vec{B}$ parallel to $d\vec{l}$

$$= \underline{\underline{\mu_0 i}}$$

(*) This result is independent of R (any radius circle would produce same result.

(*) Even if we had chosen a non-circular path, the result would have been identical

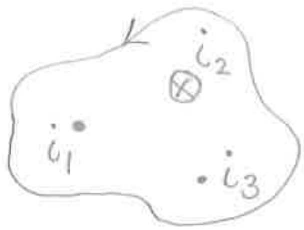


Imaginary "Amperian loop" analogous to Gaussian surface

$$\boxed{\oint \vec{B}\cdot d\vec{l} = \mu_0 i_{enc}}$$ Ampere's Law

(*) Note: Loop must be counterclockwise to get correct sign.

Example: What is the line integral of $\vec{B}\cdot d\vec{l}$ for the loop below?



At all points along this path we could compute $\vec{B}$ separately from the 3 currents and then integrate.

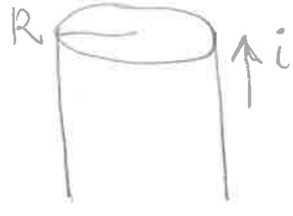
But that is too complicated!

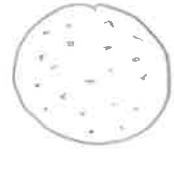
Instead, $\oint \vec{B} \cdot d\vec{l} = \mu_0 (i_1 + i_3 - i_2)$

⊛ Therefore, from Ampere's Law we can easily calculate a <u>very</u> <u>complicated</u> quantity $\oint \vec{B} \cdot d\vec{l}$, but who cares?

⊛ Just like with Gauss's Law, if symmetry let's us take B <u>outside</u> of integral, then we can compute a <u>more</u> <u>useful</u> quantity: the magnetic field.

<u>Example</u>: Consider a very thick current carrying wire of cross sectional radius R and uniform current i. What is the magnetic field <u>inside</u> and <u>outside</u> the wire?





(overhead)

From Ampere's Law, we need to calculate the current enclosed by an Amperian loop

Amperian loop of radius r encloses some of i but not all.

How much?

Recall current density definition:

$$J = \frac{i}{A}$$

$$J = \frac{i}{\pi R^2}$$

$$\Rightarrow i_{enc} = J A_{enc}$$

$$= J(\pi r^2)$$

$$= \frac{i}{\pi R^2}(\pi r^2)$$

$$\boxed{i_{enc} = i\frac{r^2}{R^2}}$$

As $r \to R$, we see that $i_{enc} \to i$ (the full current)

$$\Rightarrow \text{Ampere's Law: } \oint \vec{B} \cdot d\vec{l} = \mu_0 i \frac{r^2}{R^2}$$

From symmetry B must be constant along path.

$$B \oint dl = \mu_0 i \frac{r^2}{R^2}$$

$$B(2\pi r) = \mu_0 i \frac{r^2}{R^2}$$

$$\Rightarrow \boxed{B = \frac{\mu_0 i}{2\pi} \frac{r}{R^2} \quad (\text{for } r < R)}$$

For $r > R$ the only change is that now $i_{enc} = i$

$$\Rightarrow \oint \vec{B} \cdot d\vec{l} = \mu_0 i_{enc}$$

$$B(2\pi r) = \mu_0 i$$

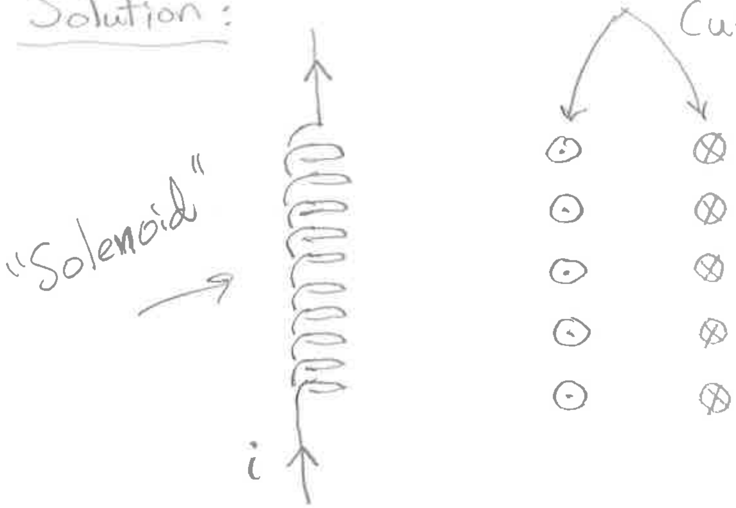
$$\boxed{B = \frac{\mu_0 i}{2\pi r} \quad \text{for } r > R}$$

this is same as for a long thin wire!

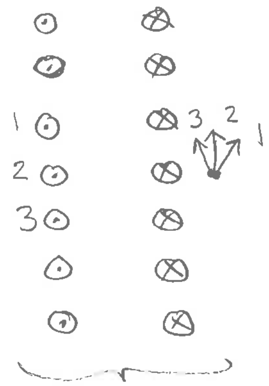
Example: What is the magnetic field inside and outside a tightly wound coil loop with N turns and length L?

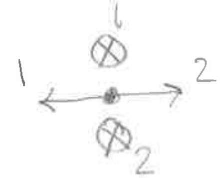
Solution:

Current directions as loops curl into and out of page.



First, $\vec{B}$ field outside is very small (difficult to prove rigorously)





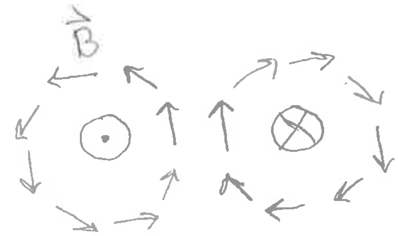
B field between also nearly 0.

Fewer currents on right-hand side contribute, but they're larger in magnitude

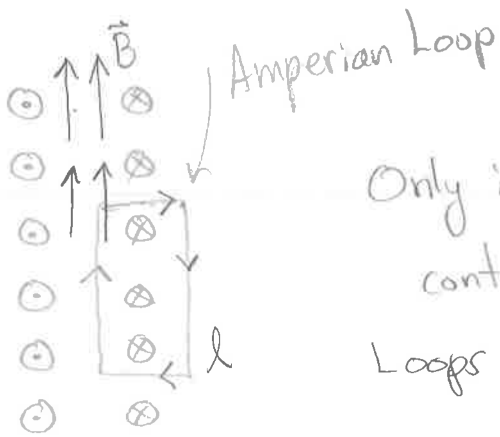
More currents on left-hand side contribute, but they're smaller in magnitude

Overall the two sets of contributions nearly cancel

$\vec{B}$ field inside:



$\vec{B}$ fields add up in center.



Only inside part of loop picks up a contribution to $\oint \vec{B} \cdot d\vec{\ell} = B\ell$.

Loops enclosed $n = \left(\frac{N}{L}\right)\ell$ ← length of Amperian Loop

↑ loop number density

$$\Rightarrow \oint \vec{B} \cdot d\ell = \mu_0 I_{enc}$$

$$B\ell = \mu_0 i \left(\frac{N}{L}\right)\ell \Rightarrow \boxed{B = \mu_0 i \frac{N}{L}}$$