PHYS 206 Lecture 1

1.1) Introduction

1.2) Position, velocity, acceleration in one dimension

1.1) Introduction

- "Mechanics" is the study of an object's motion in the presence of forces.

- In "physics" our goal is to uncover the rules and fundamental laws of nature that govern seemingly different types of motion.

Example: We are all familiar with the concept of speed.

However, we find that a much more useful quantity is velocity (a vector quantity that also includes direction of motion)

Later, we find that an even more useful concept is momentum (mass × velocity)

⊛ By uncovering the <u>laws of nature</u>, we can solve very complicated problems rather easily

<u>Types of motion we will study in this course</u>

(1) One-dimensional motion ("Linear" or "1D"): drive car down straight road, drop an egg straight down, ...

(2) Two-dimensional motion ("2D"): throw an object or fire a gun (projectile motion), Earth orbiting Sun (circular motion), ...

(3) Three-dimensional motion ("3D"): solid rotating objects, ...

Essentially, this entire course boils down to a few simple ideas:

(a) If we know all of the forces acting on an object (together with its initial location and velocity), we can exactly determine its subsequent motion at all times.

(b) Conversely, if we know an object's motion in time, we can determine exactly the net force on the object at all times.

(c) But life is often not so kind ... we might not know all of the forces: in this case we can take advantage of poweful conservation laws (momentum and energy) to predict an object's motion.

What mathematical tools and concepts do we need to know?

(1) Calculus (how to take simple derivatives and compute simple integrals)

(2) Vector algebra (how to add vectors, how to take simple derivatives of vector quantities)

1.2) Position, velocity, and acceleration in one dimension

(*) There are different ways to describe an object's motion:

Position:	$x(t)$ "1D"	or	$\vec{x}(t)$	"2D" or "3D"
Velocity:	$v(t)$	or	$\vec{v}(t)$	
Acceleration:	$a(t)$	or	$\vec{a}(t)$	

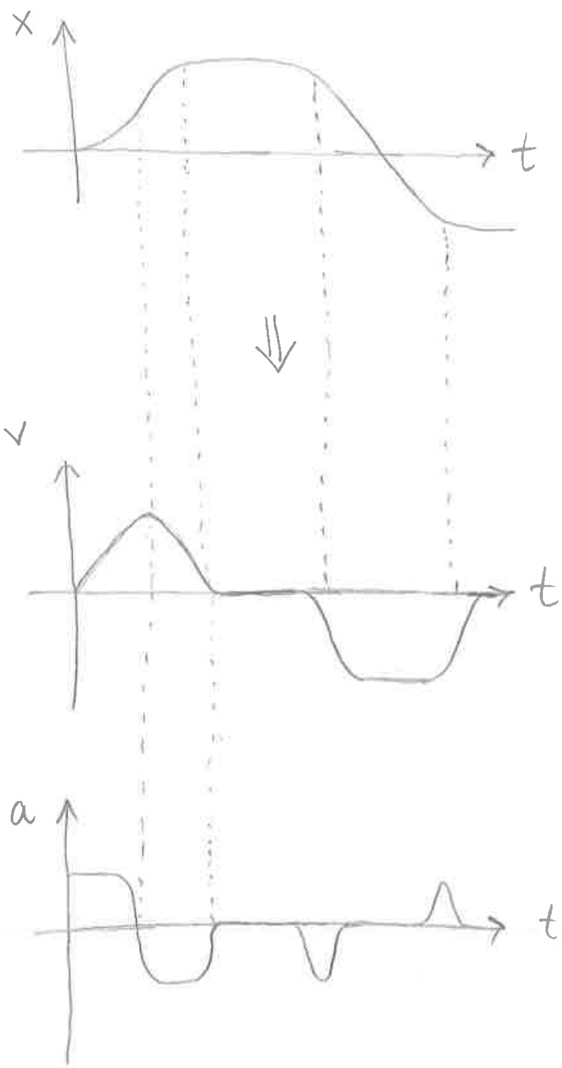
Vector quantities

Rigorous mathematical relations (one-dimensional motion)

$$v(t) = \frac{d}{dt} x(t) \qquad a(t) = \frac{d}{dt} v(t) = \frac{d^2}{dt^2} x(t)$$

⊛ If we know $x(t)$, the object's location in time, we can exactly determine its velocity and acceleration

Example:



Example: Suppose an object's position in time is given by

$x(t) = -a + bt - ct^2$. What is the object's velocity at

time $t = 10$ sec ?

(Solution) $v(t) = \frac{dx}{dt} = b - 2ct$

$$\Rightarrow v(10) = b - 2c(10) = \boxed{b - (20\,\text{sec})\,c}$$

How far has the object traveled from its starting position in that time?

(Solution) Be careful here ... the answer is not

$x(10) = -a + (10\text{sec})b - (100\,\text{sec}^2)\,c$ X this is the location at $t = 10$ sec.

Instead, the distance traveled is $x(10) - x(0)$, which is the final position minus the original position.

Then $x(10) - x(0) = -a + (10\text{sec})b - (100\text{sec}^2)c - (-a)$

$$\boxed{= (10\text{sec})\,b - (100\text{sec}^2)\,c}$$

(*) We could likewise calculate the object's acceleration at any point in time.

Main Point

$$
\begin{tikzcd}[column sep=huge]
\boxed{x(t)} \arrow[r, "\frac{d}{dt}", "\text{unique}"'] \arrow[rr, bend right, "\frac{d^2}{dt^2}", "\text{unique}"'] & \boxed{v(t)} \arrow[r, "\frac{d}{dt}", "\text{unique}"'] & \boxed{a(t)}
\end{tikzcd}
$$

Question: What if we are given instead $v(t)$ or $a(t)$...

can we uniquely determine $x(t)$?

Answer: No!

But if we are given just a little more information, we can determine $x(t)$.

(*) Recall that the "inverse operation" to differentiation is integration

Therefore, $x(t) = \int v(t)\,dt + C_1$

$$v(t) = \int a(t)\,dt + C_2$$

(*) Everytime we take an integral, we need one extra piece of information

Example: If you are given $v(t)$ and an object's starting location $x(0)$, then you can find $x(t)$ for all times.

(*) Actually, any location at a specific time is sufficient:

$x(0)$, $x(t_f)$, $x(t_r)$, ...

initial position; final position; some random intermediate position

Concrete illustration (Exam 1, 2017, Problem 1)

1. (25 Points) A car is moving along a straight line defined to be the positive x direction. Its velocity is measured and found to be a function of time given by

$$v_x(t) = \alpha t^2$$

where α is a known constant. The car was at the point $x = A$ at the time $t = 2sec$. Find the car's position as a function of time. How fast would the car be going just before it hits a wall located at $x = L$?

(Solution) Note that we are given $v(t)$ and $x(2) = A$.

$\Rightarrow$ We now can derive complete information on its motion:

$$x(t) = \int v(t)\,dt + C$$

$$x(t) = \int \alpha t^2\,dt + C = \frac{1}{3}\alpha t^3 + C$$

Since $x(2) = A \Rightarrow x(2) = \frac{1}{3}\alpha(2)^3 + C = A$

$$\Rightarrow \frac{8}{3}\alpha + C = A$$

$$\Rightarrow C = A - \frac{8}{3}\alpha$$

$$\Rightarrow \boxed{x(t) = \frac{1}{3}\alpha t^3 + A - \frac{8}{3}\alpha}$$

How fast is the car going when it reaches $x = L$?

Since we know $v(t)$ for all times we just need to find the time at which $x = L$. This is found by solving

$$x(t_f) = L$$

$$\frac{1}{3}\alpha t_f^3 + A - \frac{8}{3}\alpha = L$$

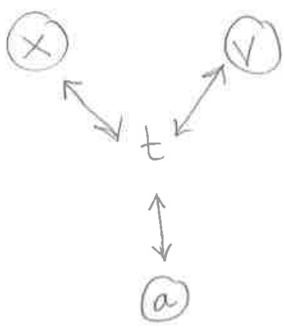
$$\Rightarrow t_f^3 = \frac{3}{\alpha}\left(L - A + \frac{8}{3}\alpha\right)$$

$$\Rightarrow \quad t_f = \left[\frac{3}{\alpha}\left(L - A + \frac{8}{3}\alpha\right)\right]^{1/3}$$

Finally, just plug this time into the expression for $v(t)$:

$$v(t_f) = \alpha t_f^2 = \alpha \left[\frac{3}{\alpha}\left(L - A + \frac{8}{3}\alpha\right)\right]^{2/3}$$

Problem Solving Tip: All of the quantities $x(t)$, $v(t)$, $a(t)$ are connected to each other by time. So, to relate x, v, a to each other, first go through time.



⊛ If we are given $a(t)$, we need one piece of information about $v(t_0)$ to get $v(t)$ for all times and one more piece of information about $x(t_0)$ to get $x(t)$.

So we can modify our previous illustration as follows

Main Point :

$$
\begin{tikzcd}
\boxed{x(t)} \arrow[r, bend left, "dx/dt"] & \boxed{v(t)} \arrow[l, bend left, "\int v(t)\,dt + C_1"] \arrow[r, bend left, "dv/dt"] & \boxed{a(t)} \arrow[l, bend left, "\int a(t)\,dt + C_2"]
\end{tikzcd}
$$

⊗ The constants of integration C_1 or C_2 are obtained from the "initial conditions", either $x(t_0)$ or $v(t_0)$.

Note that "initial" is standard terminology, but it doesn't need to actually refer to the "starting" time. The time t_0 can be anything.

Most important Derivatives/Integrals needed:

$$f(t) = at^n$$

(1) $\dfrac{df}{dt} = a\left(nt^{n-1}\right) = ant^{n-1}$

(2) $\displaystyle\int f(t)\,dt = \frac{a}{n+1}\,t^{n+1} + C$

⊗ Simple Rules: $\dfrac{d}{dt}\left(af(t) + bg(t)\right) = a\dfrac{df}{dt} + b\dfrac{dg}{dt}$

$$\frac{d}{dt}\left(f(t)g(t)\right) = \frac{df}{dt}g + f\frac{dg}{dt}$$

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$$\int \left(af(t)+bg(t)\right)dt = a\int f(t)dt + b\int g(t)dt$$

That is all of the calculus that we will need in this course.

Acceleration

* Given $a(t)$ and appropriate initial conditions, we can determine $v(t)$ and $x(t)$ at all times.

Special Case: Constant Acceleration $a(t) = c$.

* This is of great practical importance because on Earth's surface all objects in freefall experience the exact same acceleration $\boxed{a(t) = g = 9.8 \text{ m/s}^2}$ toward the ground

Since we are considering only one-dimensional motion for now, that means (a) dropping an object, (b) tossing an object straight upward, etc.

* For constant acceleration a we can derive 3 Kinematic equations:

(1) $v(t) = v_0 + at$ Initial condition: v_0

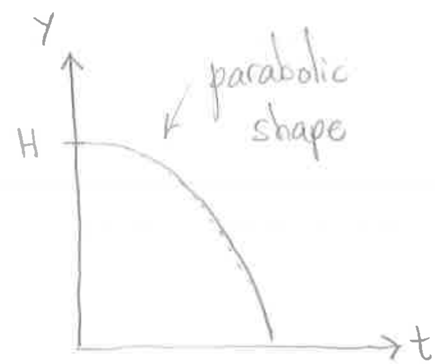
(2) $x(t) - x_0 = v_0 t + \frac{1}{2}at^2$ Initial conditions: v_0, x_0

(3) $v^2(t) = v_0^2 + 2a\left(x(t) - x_0\right)$ Initial conditions: v_0, x_0

* Probably best to memorize these. Deriving them can result in tedious algebra.

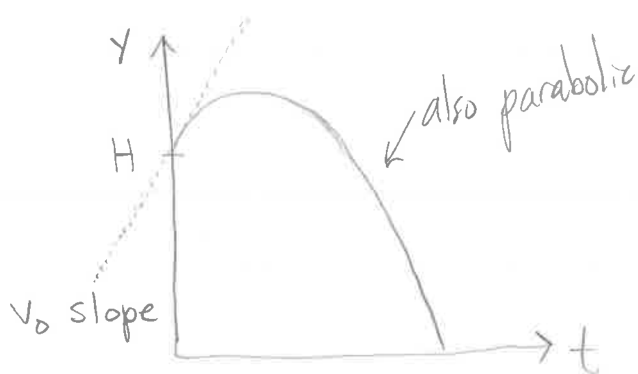
What does motion under constant acceleration look like graphically?

Example: Drop egg from height H.



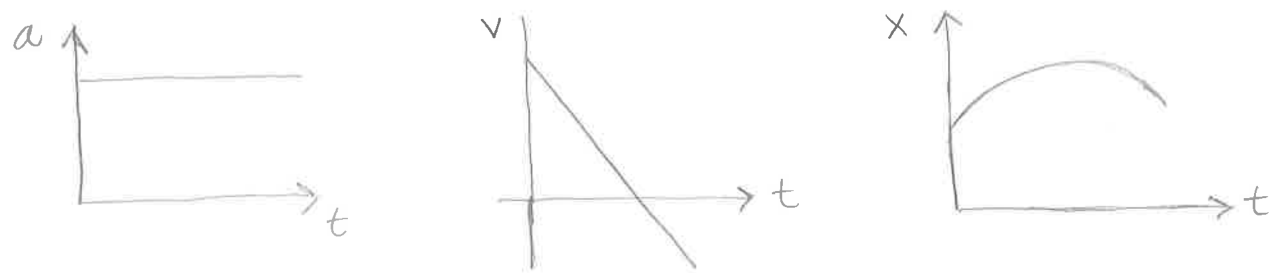
$$y(t) = -\frac{1}{2}gt^2 + H$$
$$= -\frac{1}{2}\left(9.8\,m/s^2\right)t^2 + H$$

Example: Toss egg from height H and initial velocity v_0



$$y(t) = v_0 t + \frac{1}{2}at^2 + H$$
$$= v_0 t + \frac{1}{2}\left(-9.8\,m/s^2\right)t^2 + H$$

Question: Which of the following plots are consistent?



(a) $a(t)$ and $v(t)$ only
(b) $v(t)$ and $x(t)$ only
(c) $a(t)$ and $x(t)$ only
(d) $a(t)$, $v(t)$, and $x(t)$
(e) None

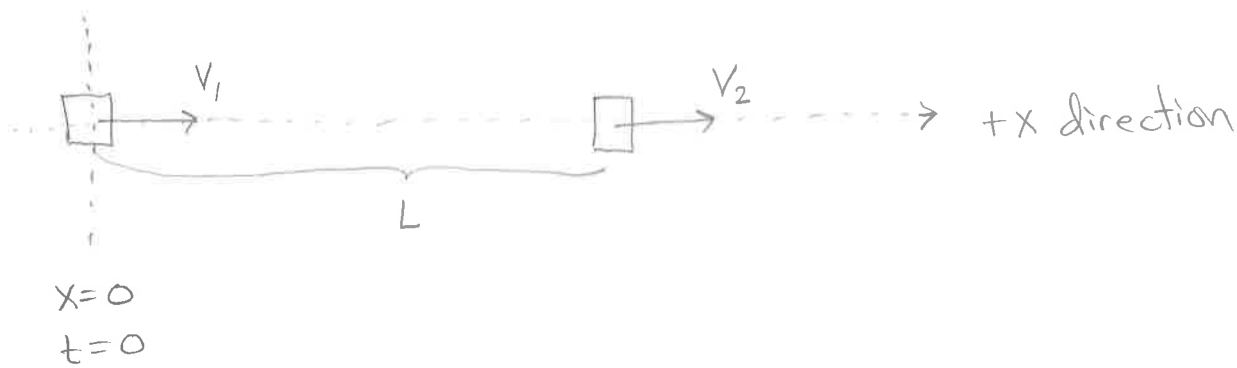
Problem: A fighter jet launches a rocket with constant acceleration $a(t) = \alpha$ toward an enemy jet that is moving directly away. If the fighter jet and enemy have constant velocities v_1 and v_2, and are separated by a distance L at the time of launch, how long does it take the rocket to hit its target?

Solution:

Step 1: Sketch a picture.

Step 2: If we don't give you the coordinate system, then define the origin and positive x-direction as you prefer.

Step 3: Determine separately $x_1(t)$, $x_2(t)$, ... for all of the objects of relevance. Their motions are independent, clearly separating them in your mind will reduce confusion.



Motion of enemy plane: $x_e(t) = \int v_e(t)\,dt + C$

$$\Rightarrow X_e(t) = V_2 t + C$$

$$(\text{at } t=0,\ x_e(t)=L) \Rightarrow X_e(0) = \cancelto{0}{V_2(0)} + C = L$$

$$\underline{X_e(t) = V_2 t + L}$$

Motion of fighter jet: who cares ?!

Motion of rocket: Hmm, do we have 2 initial conditions ?

Yes, $x_r(0) = 0$

$V_r(0) = V_1 \leftarrow$ the initial velocity of rocket is not 0 (it travels with plane at time of launch)

Therefore, $x_r(t) = v_1 t + \frac{1}{2}\alpha t^2 + \cancel{x_r(0)}^{\,0}$

Now that we know $x_e(t)$ and $x_r(t)$ for all times, we can calculate the specific time t_0 at which $x_e(t_0) = x_r(t_0)$:

$$v_2 t_0 + L = v_1 t_0 + \frac{1}{2}\alpha t_0^2$$

$$\Rightarrow \frac{1}{2}\alpha t_0^2 + (v_1 - v_2) t_0 - L = 0$$

$$\Rightarrow t_0^2 + \frac{2}{\alpha}(v_1 - v_2) t_0 - \frac{2L}{\alpha} = 0$$

main result

$$\Rightarrow t_0 = \frac{v_2 - v_1}{\alpha} + \frac{1}{2}\sqrt{\frac{4}{\alpha^2}(v_1 - v_2)^2 + \frac{8L}{\alpha}}$$

$$= \frac{v_2 - v_1}{\alpha} + \frac{1}{\alpha}\sqrt{(v_1 - v_2)^2 + 2\alpha L}$$

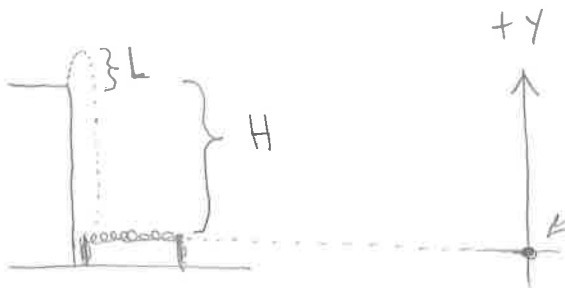
Final algebra, not worth many points on exam

(*) Whenever there are multiple objects in motion, ask yourself whether their motions are independent or somehow directly connected

→ If they are independent, the problem will break down into several easier ones

Question: Suppose a stuntman jumps from the top of a building a height H above a safety net at ground level. The person's jump takes him a vertical distance L above the roof. What is his velocity when he hits the net? (Neglect air resistance and any horizontal motion.)

Solution



Convenient to choose $y=0$ at level of safety net.

We are only explicitly given two pieces of information (2 heights). But there are two implicit pieces of information

(a) acceleration $a = -g$ (minus sign important: the person accelerates downward, which is opposite our choice of +y direction)

(b) $v(y = H+L) = 0$ (since at maximum height the velocity must be 0) which of the kinematic equations (for constant acceleration motion)

can we employ?

(1) $v(t) = v_0 + at$ ✗ we don't know time

(2) $x(t) = v_0 t + \frac{1}{2}at^2 + x_0$ ✗ we don't know time

(3) $v^2(t) = v_0^2 + 2a(x(t) - x_0)$ Yes

$$v^2(t_0) = 0^2 + 2(-g)(0 - [H+L])$$

minus important

final position is at net level

$$\Rightarrow \boxed{v = \sqrt{2g(H+L)}}$$

Focus Question: 2010, Exam 1, Problem 1

You should be able to solve the following problems from previous Exam 1's:

2005 (1)	2008 (1)	2012 (1a)
2006 (1a)	2009 (1)	2012 (3)
2006 (2)	2010 (1)	2013 (2)
2007 (1)	2011 (1)	2014 (2)