USEFUL EQUATIONS

If $f(x) = a\,x^n$, then

$$\frac{df}{dx} = n\,a\,x^{n-1} \qquad \text{and} \qquad \int f(x)\,dx = \frac{a}{n+1}\,x^{n+1} + C$$

Motion under <u>constant</u> acceleration a:

$$v(t) = at + v(0), \qquad x(t) = \frac{1}{2}at^2 + v(0)t + x(0), \qquad v^2(t_2) - v^2(t_1) = 2a\Big[x(t_2) - x(t_1)\Big]$$

Work – Kinetic Energy Theorem:

$$\int_{\vec{r}_1}^{\vec{r}_2} \vec{F}_{tot} \cdot d\vec{r} = \frac{1}{2}mv^2(\vec{r}_2) - \frac{1}{2}mv^2(\vec{r}_1).$$

If $\vec{F}$ is conservative, then there exists a potential energy function U such that

$$U(\vec{r}_2) - U(\vec{r}_1) = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r}, \qquad F_x = -\frac{\partial U}{\partial x}, \qquad F_y = -\frac{\partial U}{\partial y}.$$

Velocity and acceleration in polar coordinates:

$$\vec{v} = \frac{dr}{dt}\hat{i}_r + r\omega\hat{i}_\theta, \qquad \vec{a} = \left(\frac{d^2r}{dt^2} - r\omega^2\right)\hat{i}_r + \left(2\omega\frac{dr}{dt} + r\alpha\right)\hat{i}_\theta, \qquad \alpha = \frac{d\omega}{dt}$$

Angular motion:

$$\vec{L} = \vec{r} \times \vec{p}, \qquad \vec{\tau} = \vec{r} \times \vec{F}, \qquad L = I\omega, \qquad I = mr^2 \text{ (point particle)}, \qquad \tau = I\alpha$$

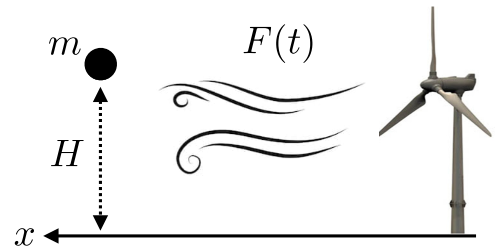
Note: The symbol g stands for the **magnitude** of the acceleration due to gravity, and therefore it is always a positive quantity.

Free-body force diagrams are very important!

Do not spend too much time on algebra!

1. (15 points) At time $t = 0$ a ball of mass m is dropped from rest at height H in a wind tunnel where in addition to gravity the ball experiences a time-dependent horizontal force $F(t) = \alpha t$, where α is a positive constant. (i) What is the speed of the ball as a function of time? (ii) How far will the object have traveled horizontally before it hits the ground?

Law



Application

+2 +1

$$v_x = \int a_x \, dt = \int \alpha t \, dt = \frac{1}{2}\alpha t^2 + \cancel{C}^{\,0}$$ +1

$$v_y = -gt$$ +2

$$\Rightarrow \quad v = \sqrt{v_x^2 + v_y^2} = \sqrt{\frac{1}{4}\alpha t^4 + g^2 t^2}$$ +1

$$y = y_0 + v_0 t + \frac{1}{2} a t^2 \quad \Rightarrow \quad 0 = H - \frac{1}{2} g t^2$$

+2 (or equivalent)

$$t^* = \sqrt{2H/g}$$ +2

+2 +1

$$\Delta x = \int \frac{1}{2}\alpha t^2 \, dt = \frac{1}{6}\alpha t^3$$

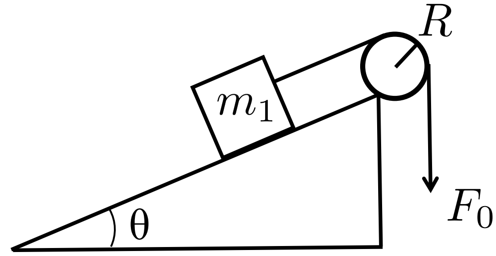
$$\hookrightarrow \frac{1}{6}\alpha t^{*3}$$

+1

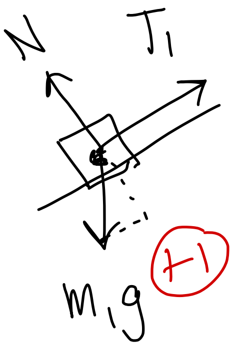
Result

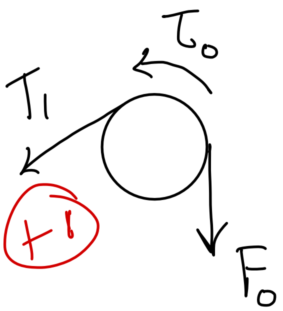
2. (15 points) A block of mass m_1 is placed on a frictionless inclined plane of angle θ. A force of magnitude F_0 is applied to the string as shown. The string is unstretchable, has negligible mass, and pulls without slipping. The pulley has radius R and moment of inertia I with respect to the axis of rotation. There is a friction force at the axle which exerts a torque of constant magnitude τ_0 opposing the rotation. (i) Write the system of equations that could be solved to find the acceleration of the block. (ii) Find the acceleration of the block.

Law



Application





$$T_1 - m_1 g\sin\theta = m_1 a \quad +3$$

$$F_0 R - T_1 R - \tau_0 = I\alpha \quad +5$$

$$a = R\alpha \quad +2$$

$$T_1 = m_1(a + g\sin\theta)$$

Some reasonable algebra +2

$$\Rightarrow F_0 R - R m_1 a - R m_1 g\sin\theta - \tau_0 = I\frac{a}{R}$$

$$\Rightarrow F_0 R - R m_1 g\sin\theta - \tau_0 = a\left(\frac{I}{R} + R m_1\right)$$

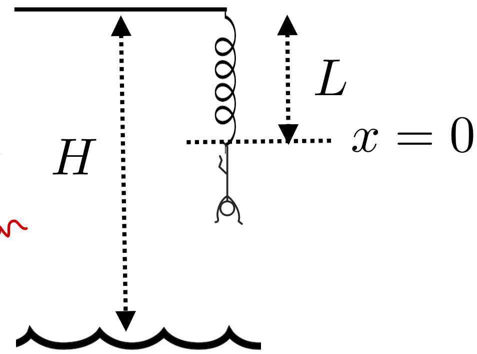
$$+1 \quad a = \frac{R^2(F_0 - m_1 g\sin\theta) - \tau_0 R}{I + m_1 R^2}$$

Result

3. (15 points) A thrill-seeking physicist of mass m bungee jumps by stepping off a bridge located a distance H above a river. Confident in her knowledge of mechanics, she wants to choose a bungee cord such that she will come to rest just as she touches the river below. The bungee cord exerts no force until the jumper has fallen a distance L and then it exerts a force $F(x) = -\alpha x^3$, where x denotes the distance stretched relative to its natural rest length. (i) What is the jumper's speed when she has fallen a distance L? (ii) What must be the value of α needed in order to stop her just before touching the water?

Law

$U_g = mgH$ (+2)



Application

$mgL = \frac{1}{2}mv^2$ (+2)

$\Rightarrow v_L = \sqrt{2gL}$ (+1)

or equiv. solution

$U = -\int F\,dx$ (+2)

$U = -\int -\alpha x^3\,dx$

$U = \frac{1}{4}\alpha x^4$ (+2)

$KE^i + U_s^i + U_g^i = KE^f + U_s^f + U_g^f$ (+2)

(different ways to set this up)

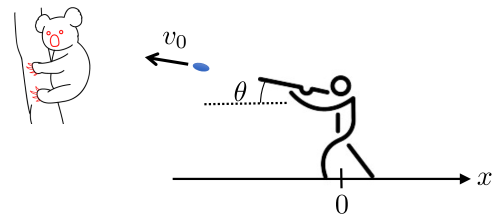
$mgL + 0 + mg(H-L) = 0 + \frac{1}{4}\alpha(H-L)^4$ (+3)

$\Rightarrow \alpha = \dfrac{4mgH}{(H-L)^4}$ (+1)

Result

4. (15 points) A student of mass M stands at the origin ($x = 0$) on a frozen lake and fires a tranquilizer dart of mass m and speed v_0 at an angle θ with respect to the horizontal toward a polar bear. The recoil from the gun causes the student to travel along the $+x$ direction, where there is a coefficient of friction $\mu(x) = \mu_0 \alpha x$ with α a constant. How far does the student travel before coming to rest?

Law



Application

$\vec{p}$ conservation +2

+1 $0 = -mv_0\cos\theta + Mv_1$ +2

$\Rightarrow v_1 = \frac{m}{M} v_0 \cos\theta$ +1

+2 W-KE: $0 - \frac{1}{2}M\left(\frac{m}{M}v_0\cos\theta\right)^2 = \int_0^{x_f} -\mu_0(1+\alpha x)\,dx$ +1 Mg

+2

+1 sign

$\Rightarrow -\cancel{\frac{1}{2}}\frac{m^2}{M}v_0^2\cos^2\theta = -\cancel{\frac{1}{2}}\alpha\mu_0 M g x_f^2$ +2

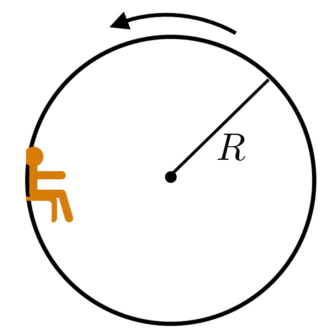
$x_f = \frac{m}{M}v_0\cos\theta\sqrt{\frac{1}{\alpha\mu_0 g}}$

+1

Result

5. (15 points) At astronaut training camp, a student of mass m is placed in a horizontal circular centrifuge (**overhead view** shown below) that starts from rest at $t = 0$ and undergoes angular acceleration $\alpha = c_1 t$, where c_1 is a constant, for a total length of time T_0. (i) What must be the radius R of the centrifuge so that at time T_0 the student experiences an acceleration five times larger than gravity ($a = 5g$)? (ii) What is the student's velocity at that time?

Law



(overhead view)

Application

+2 +2 +1

$$\omega(t) = \int c t \, dt = \frac{1}{2} c t^2 + \cancel{c_1}^{\,0}$$

$$\vec{a} = \left(\frac{d^2 r}{dt^2} - r\omega^2\right)\hat{\imath}_r + \left(r\alpha + 2\omega\frac{dr}{dt}\right)\hat{\imath}_\theta \quad +2$$

magnitude $\sqrt{a_r^2 + a_\theta^2}$ +2

$$= -R\frac{1}{4}c^2 t^4\,\hat{\imath}_r + Rct\,\hat{\imath}_\theta \quad +2$$

$$\Rightarrow a = \sqrt{\frac{1}{16}R^2 c^4 t^8 + R^2 c^2 t^2} = 5g$$

$$= R\sqrt{\frac{1}{16}c^4 t^8 + c^2 t^2} = 5g$$

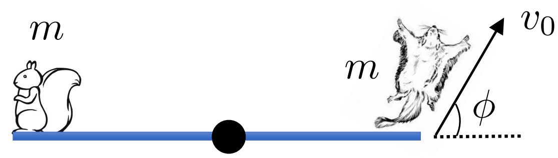
$$\Rightarrow R = \frac{5g}{cT_0\sqrt{\frac{1}{16}c^2 T_0^6 + 1}} \quad +1$$

Result

$$\underbrace{V = R\omega}_{+2} = \underbrace{\frac{1}{2}cRT_0^2}_{+1}$$

6. (15 points) Two squirrels with equal mass m sit at rest on the left and right ends of a bar with moment of inertia I_b and total length $2L$ that is free to rotate horizontally about a centered frictionless axle (**overhead view** shown below). At time $t = 0$, the right squirrel jumps off with speed v_0 at the angle ϕ. (i) What will be the angular velocity of the left squirrel and bar after the right squirrel has jumped off? (ii) What is the angular momentum (**magnitude and direction**) of each squirrel relative to the rotation center of the bar after the right squirrel has jumped?

Law



(overhead view)

Application

$\vec{L}$ conservation (+2)

(+2) Definition

$$\vec{L}_i = \vec{r}\times\vec{p} = \overbrace{Lmv_0}^{+1}\,\overbrace{\sin\phi}^{+1}$$

(+2) Definition

$$L_f = I\omega = \overbrace{(I_b + mL^2)}^{+1}\,\omega_f \quad (+1)$$

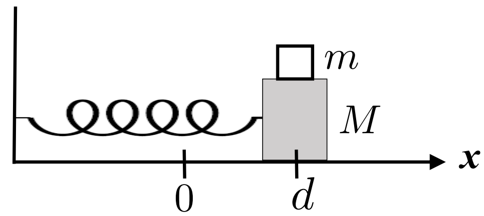
$$\omega_f = \frac{Lmv_0}{I_b + mL^2} \quad (+1)$$

$$\vec{L}_r = Lmv_0\sin\phi \quad \odot \quad (+1)$$

$$\vec{L}_l = \underbrace{(mL^2)}_{+1}\,\underbrace{\frac{Lmv_0}{I_b + mL^2}}_{+1} \quad \otimes \quad (+1)$$

Result

7. (15 points) A block of mass M lies on a frictionless surface and is connected to an ideal spring that produces a force $F(x) = -kx$. A second block of mass m is placed on top of the first block and both are pulled so that the spring is extended a distance d from its natural rest length. The blocks are released from rest at time $t = 0$. (i) What will be the velocity of the blocks as a function of time if the top block does not slide relative to the lower block at any time? (ii) What must be the minimum coefficient of friction μ between the two blocks so that the top block does not slide relative to the lower block at any time?



Law

+2

$\vec{F} = m\vec{a}$

Application

+2

$$\vec{F} = -K\vec{x} = (m+M)\frac{d^2\vec{x}}{dt^2}$$

+2

+2 General solution

$$x(t) = A\cos(\omega t) + B\sin(\omega t), \quad \omega = \sqrt{\frac{K}{m+M}}$$

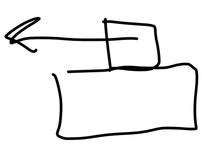
$$x(0) = d \qquad v(0) = 0$$

$$\Rightarrow d = A$$ +1

+1

$$v(t) = -A\omega\sin(\omega t) + B\omega\cos(\omega t) \Rightarrow v(0) = 0 = B$$

$$\underline{v(t) = -d\omega\sin(\omega t)}$$ +1



+1

$$Kd = (m+M)a_{max} \Rightarrow a_{max} = \frac{Kd}{m+M}$$

Result

+1 +1 +1

$$\Rightarrow F_\mu = m a_{max} = \mu m g \Rightarrow \mu = \frac{Kd}{g(m+M)}$$