Example (Exam 2, 2014, Q4)

A spherical shell has inner radius A, outer radius B, and resistivity ρ_0. It is surrounded by a second spherical shell with inner radius B and outer radius G. The resistivity of the outer conductor is $\rho(r) = \rho_0 \frac{r^2}{G^2}$. The ⊕ terminal of a battery is connected at A, and the ⊖ terminal is first connected at B and then at G. Find the two currents that flow if the battery voltage is V.

Solution: Follow 5 step process.

① Direction of current flow is radially outward

② Cross-sectional area perpendicular to current flow is

$$A = 4\pi r^2 \text{ (surface area of sphere)}$$

③ Current density is $J = \frac{i}{A} = \frac{i}{4\pi r^2}$

④ From Ohm's Law, $\vec{E} = \rho \vec{J} = \rho_0 \frac{i}{4\pi r^2} \hat{i}_r$ (case 1)

⑤ Finally, $V(A) - V(B) = -\int_B^A \vec{E} \cdot d\vec{r} = -\int_B^A \rho_0 \frac{i}{4\pi r^2} dr$

$$= -i \frac{\rho_0}{4\pi} \left[-\frac{1}{r} \right]_B^A = i \frac{\rho_0}{4\pi} \left[\frac{1}{A} - \frac{1}{B} \right] = V$$

$$\Rightarrow \boxed{i = \frac{4\pi}{\rho_0} \left(\frac{AB}{B-A} \right) V}$$

In the second setup, we proceed the same way:

① current radially outward

② $A = 4\pi r^2$

③ $J = \dfrac{i}{4\pi r^2}$

④ $\vec{E} = \rho \vec{J} = \begin{cases} \dfrac{\rho_0 i}{4\pi r^2} & (A<r<B) \\ \rho_0 \dfrac{r^2}{G^2} \dfrac{i}{4\pi r^2} & (B<r<G) \end{cases}$

Note that i is same through both parts of resistor.

⑤ $\Delta V = V(A) - V(G) = -\int_G^A \vec{E}\cdot d\vec{r}$

same as previously

$$= -\int_G^B \rho_0 \frac{i}{4\pi G^2}\,dr - \int_B^A \rho_0 \frac{i}{4\pi r^2}\,dr$$

$$= -\frac{\rho_0}{4\pi G^2}(B-G)\,i - \frac{\rho_0}{4\pi}\frac{B-A}{AB}\,i = V$$

$$\Rightarrow i\,\frac{\rho_0}{4\pi}\left[\frac{G-B}{G^2} + \frac{B-A}{AB}\right] = V$$

$$\Rightarrow i = \frac{4\pi}{\rho_0} V \left[\frac{G-B}{G^2} + \frac{B-A}{AB}\right]^{-1}$$

$$\rightarrow \frac{4\pi}{\rho_0} V \left(\frac{4A-2A}{16A^2} + \frac{2A-A}{2A^2}\right)^{-1} = \boxed{\frac{4\pi}{\rho_0} V \left(\frac{8A}{5}\right)}$$