Example (Exam 2, 2015, Q3)

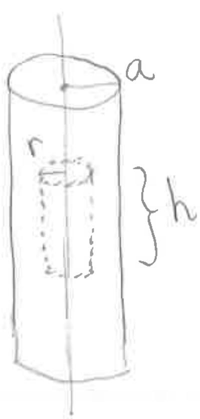
A very long cylinder has radius a and length L. It has a total charge Q spread throughout its volume uniformly. It is inside a hollow perfectly conducting cylinder of length L, inner radius b, and outer radius f. Find $V(D) - V(0)$, where $D > f$.

Solution: Strategy is to use Gauss's Law to determine $\vec{E}(r)$ and then $V(D) - V(0) = -\int_0^D \vec{E} \cdot d\vec{r}$ to get potential difference.

(*) Need to find $\vec{E}$ everywhere

(a) $E(r<a)$:

① Draw Gaussian surface G that passes through r (length will not matter, but choose h)



From symmetry $\oint_G \vec{E} \cdot d\vec{A} = (2\pi r h) E$

(area) × (length)

$\pi a^2 \times L$

② To get q_{enc}, first determine $\rho = \dfrac{Q}{V} = \dfrac{Q}{\pi a^2 L}$.

Then $q_{enc} = \int \rho \, dV = \dfrac{Q}{\pi a^2 L} \int dV$

Integrate over Gaussian surface

radius and length of physical cylinder not Gaussian surface

$$\Rightarrow q_{enc} = \frac{Q}{\pi a^2 L} \int_0^{r} \underbrace{2\pi r h \, dr}_{\text{Volume of Gaussian cylinder}}$$

radius of Gaussian cylinder, not "a".

$$V = \pi r^2 h$$

$$\Rightarrow \frac{dV}{dr} = 2\pi r h \Rightarrow dV = 2\pi r h \, dr$$

$$\Rightarrow q_{enc} = \frac{Q}{\pi a^2 L}\left(2\pi h \frac{1}{2} r^2\right) = Q \frac{h}{L}\frac{r^2}{a^2}$$

$$\Rightarrow \oint \vec{E}\cdot d\vec{A} = \frac{1}{\varepsilon_0} q_{enc}$$

$$\Rightarrow 2\pi r \cancel{h} E = \frac{1}{\varepsilon_0} Q \frac{\cancel{h} r^2}{L a^2}$$

$$\Rightarrow \boxed{E = \frac{Q}{2\pi\varepsilon_0}\frac{r}{L a^2}}$$

(b) $E(a<r<b)$: Same Gaussian surface, just larger r.

$$\oint \vec{E}\cdot d\vec{A} = E(2\pi r h)$$

$$q_{enc} = \frac{Q}{\pi a^2 L}\int_0^a 2\pi r h \, dr = \frac{Q}{\pi a^2 L}\left(\pi a^2 h\right)$$

$$= Q\frac{h}{L}$$

$$\Rightarrow \boxed{E = \frac{1}{2\pi\varepsilon_0}\frac{Q}{L}\frac{1}{r}}$$

(c) $\underline{E(b<r<f)} = 0$ (inside conductor)

(d) $\underline{E(r>f)}$: $\oint \vec{E}\cdot d\vec{A} = E(2\pi r h)$ same

q_{enc} = same as in part (b) since the conductor contains no net charge

$$\Rightarrow E = \frac{1}{2\pi\varepsilon_0}\frac{Q}{L}\frac{1}{r}$$

Finally $V(D) - V(0) = -\int_0^D \vec{E}\cdot d\vec{r}$

$$= -\int_0^a \left(\frac{Q}{2\pi\varepsilon_0}\frac{r}{La^2}\right)dr - \int_a^b \left(\frac{1}{2\pi\varepsilon_0}\frac{Q}{L}\frac{1}{r}\right)$$

$$-\int_f^D \left(\frac{1}{2\pi\varepsilon_0}\frac{Q}{L}\frac{1}{r}\right)$$

$$= \frac{-Q}{2\pi L\varepsilon_0}\left[\frac{1}{a^2}\int_0^a r\,dr + \int_a^b \frac{1}{r}dr + \int_f^D \frac{1}{r}dr\right]$$

$$= \frac{-Q}{2\pi L\varepsilon_0}\left[\frac{a^2}{2a^2} + \ln(b/a) + \ln(D/f)\right]$$

$$= \frac{-Q}{4\pi\varepsilon_0 L}\left[1 + 2\ln\left(\frac{bD}{af}\right)\right]$$

For $b = 2a$ and $D = 2f$ $\Rightarrow \Delta V = \frac{-Q}{4\pi\varepsilon_0 L}\left[1 + 4\ln(2)\right]$