EXAM III Physics 206 FALL 2020

Last Name.. First.................................... Section Number..........

USEFUL EQUATIONS

If $f(x) = a\,x^n$, then

$$\frac{df}{dx} = n\,a\,x^{n-1}$$

$$\int f(x)\,dx = \frac{a}{n+1}\,x^{n+1} + C$$

Work – Kinetic Energy Theorem:

$$\int_{\vec{r}_1}^{\vec{r}_2} \vec{F}_{tot} \cdot d\vec{r} = \frac{1}{2}mv^2(\vec{r}_2) - \frac{1}{2}mv^2(\vec{r}_1).$$

If $\vec{F}$ is conservative, then there exists a potential energy function U such that

$$U(\vec{r}_2) - U(\vec{r}_1) = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r}$$

and

$$F_x = -\frac{\partial U}{\partial x}, \quad F_y = -\frac{\partial U}{\partial y}.$$

$$\vec{L} = \vec{r} \times \vec{p}, \quad \vec{\tau} = \vec{r} \times \vec{F}$$

Moment of inertia:

$$I = L/\omega, \quad I = mr^2 \text{ (point particle)}$$

Note: The symbol g stands for the **magnitude** of the acceleration due to gravity, and therefore it is always a positive quantity.

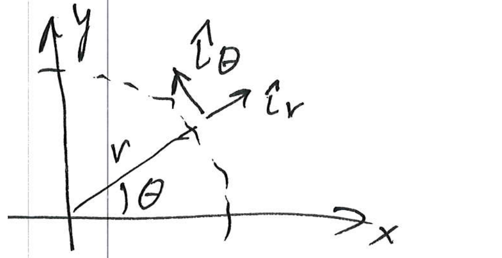
Free-body force diagrams are very important!

Do not spend too much time on algebra!

1. (25 points) Derive the expressions for velocity and acceleration in polar coordinates (that is, in terms of the unit vectors $\hat{i}_r$ and $\hat{i}_\theta$).

Law

Application



$$\vec{r} = r\,\hat{i}_r \qquad \hat{i}_r = \cos\theta(t)\,\hat{i} + \sin\theta(t)\,\hat{j}$$

$$\hat{i}_\theta = -\sin\theta(t)\,\hat{i} + \cos\theta(t)\,\hat{j}$$

$$\frac{d\hat{i}_r}{dt} = -\sin\theta\,\frac{d\theta}{dt}\,\hat{i} + \cos\theta\,\frac{d\theta}{dt}\,\hat{j} = \frac{d\theta}{dt}\left(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\right) =$$

$$= \frac{d\theta}{dt}\,\hat{i}_\theta$$

$$\frac{d\hat{i}_\theta}{dt} = -\cos\theta\,\frac{d\theta}{dt}\,\hat{i} - \sin\theta\,\frac{d\theta}{dt}\,\hat{j} = -\frac{d\theta}{dt}\,\hat{i}_r$$

$$\frac{d\vec{r}}{dt} = \frac{dr}{dt}\,\hat{i}_r + r\,\frac{d\hat{i}_r}{dt} = \frac{dr}{dt}\,\hat{i}_r + r\,\frac{d\theta}{dt}\,\hat{i}_\theta$$

$$\frac{d^2\vec{r}}{dt^2} = \frac{d^2r}{dt^2}\,\hat{i}_r + \frac{dr}{dt}\,\frac{d\hat{i}_r}{dt} + \frac{dr}{dt}\,\frac{d\theta}{dt}\,\hat{i}_\theta + r\,\frac{d^2\theta}{dt^2}\,\hat{i}_\theta + r\,\frac{d\theta}{dt}\,\frac{d\hat{i}_\theta}{dt} =$$

$$= \frac{d^2r}{dt^2}\,\hat{i}_r + \frac{dr}{dt}\,\frac{d\theta}{dt}\,\hat{i}_\theta + \frac{dr}{dt}\,\frac{d\theta}{dt}\,\hat{i}_\theta + r\,\frac{d^2\theta}{dt^2}\,\hat{i}_\theta - r\left(\frac{d\theta}{dt}\right)^2\hat{i}_r =$$

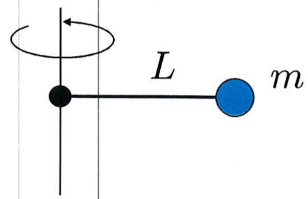
$$= \hat{i}_r\left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right) + \hat{i}_\theta\left(2\,\frac{dr}{dt}\,\frac{d\theta}{dt} + r\,\frac{d^2\theta}{dt^2}\right)$$

Result

2. (25 points) A ball of mass m is superglued to the end of a metal rod of length L that rotates horizontally about a vertical axle. At time $t = 0$ the ball and rod start from rest and undergo angular acceleration $\alpha = c_1 t$, where c_1 is a constant. If the ball breaks free at time t_b, what is the maximum force that the glue can hold? **Neglect the small effects due to gravity.**

Law

3 $\vec{F} = m\vec{a}$ 3 $a = \sqrt{a_r^2 + a_\theta^2}$



Application

$\omega = \int \alpha \, dt = \int c_1 t \, dt =$ 3 1

$= \frac{c_1 t^2}{2} + c$ 2 $\omega(0) = 0 \rightarrow c = 0$ 1

$\vec{a} = \left(\frac{d^2 r}{dt^2} - r\omega^2\right)\hat{i}_r + \left(2\frac{dr}{dt}\omega + r\frac{d^2\theta}{dt^2}\right)\hat{i}_\theta =$ 3

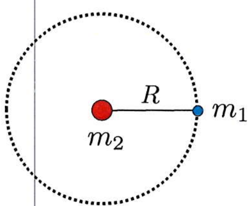
$= -\left(L\left(\frac{c_1 t^2}{2}\right)^2\right)\hat{i}_r + L \cdot c_1 t \, \hat{i}_\theta$ 3 3

2 $t = t_b$ $F_{max} = m\sqrt{L^2\left(\frac{c_1 t_b^2}{2}\right)^4 + L^2 c_1^2 t_b^2} = mLc_1 t_b \sqrt{1 + \frac{c_1^2 t_b^6}{16}}$ 1

IF no a_θ at all, -7

Result

3. (25 points) A satellite of mass m_1 rotates around Planet X with mass m_2 in a circular orbit. The magnitude of the gravitational force is $F = G\frac{m_1 m_2}{r^2}$, where G is a positive constant and r is the distance between the two masses. (i) What speed must the satellite have in order to orbit at a radius R? (ii) If at time $t = 0$, the satellite uses thrusters to move from the position $(r = R, \theta = 0)$ to $(r = 2R, \theta = 90°)$, how much work is done by gravity along the satellite's path?



Law

3 $W = \int_{R_1}^{R_2} F_r\,dr + \int_{\theta_1}^{\theta_2} F_\theta\, r\,d\theta$ 3 $F = ma$

Application

2 1 3

$$G\frac{m_1 m_2}{r^2} = m_1 r \omega^2$$

2

$$\omega^2 = G\frac{m_2}{r^3} \qquad V = R\omega = R\sqrt{G\frac{m_2}{R^3}} = \sqrt{G\frac{m_2}{R}}$$ 1

1 2 2 1 2

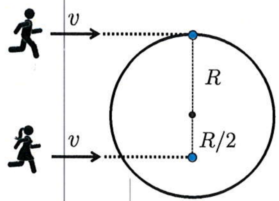
$$W = \int_R^{2R} F_r\,dr = \int_R^{2R} -G\frac{m_1 m_2}{r^2}\,dr = +G\frac{m_1 m_2}{r}\Big|_R^{2R} =$$

1

$$= -G\frac{m_1 m_2}{2R}$$ 1

Result

4. (25 points) A boy and girl with the same mass m run with the same speed v toward a stationary merry-go-round of radius R that is free to rotate around a frictionless axle through its center. The merry-go-round has moment of inertia I_0. (i) If the two students run horizontally and land at the distances R and $R/2$ as shown in the figure, what will be the magnitude of the angular velocity of the merry-go-round after the students land? (ii) If the girl then walks ~~radially~~ in a straight line to the boy at $r = R$, what is the final angular velocity of the merry-go-round?



Law

2 2

$$\vec{L} = \vec{r} \times m\vec{v} \qquad \vec{L} = I\vec{\omega}$$

Application

2 2 2

(I) $$L_i = +mv\frac{R}{2} - mvR$$

1 1 1 1

$$L_f = \left(I + mR^2 + m\left(\frac{R}{2}\right)^2\right)\omega$$

4

$$L_i = L_f$$

$$\omega = \frac{-mv\frac{R}{2}}{I + \frac{5}{4}mR^2}$$

1

1 1 1 2

(II) $$L_f' = \left(I + 2mR^2\right)\omega' = -mv\frac{R}{2}$$

$$\omega' = \frac{-mv\frac{R}{2}}{I + 2mR^2}$$

1

Result