Exam 3 Review

4 Types of Questions to expect:

(1) Derive $\vec{v}$ and $\vec{a}$ in polar coordinates

(2) "Inverse Newton's Laws" $\longrightarrow$ From the motion, find the force (or torque, etc.)

(3) Work and potential energy for radial forces

(4) Conservation of angular momentum

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(1) $\vec{v}$ and $\vec{a}$ in polar coordinates

Can start by assuming only 3 things:

(1) $\vec{r} = r\hat{\imath}_r$

(2) $\hat{\imath}_r = \cos\theta\,\hat{\imath}_x + \sin\theta\,\hat{\imath}_y$

(3) $\hat{\imath}_\theta = -\sin\theta\,\hat{\imath}_x + \cos\theta\,\hat{\imath}_y$

Must derive $\frac{d}{dt}(\hat{\imath}_r) = \frac{d\theta}{dt}\hat{\imath}_\theta$ and

$$\frac{d}{dt}(\hat{\imath}_\theta) = -\frac{d\theta}{dt}\hat{\imath}_r$$

using the derivative <u>chain rule</u>.

From those two relations, calculate

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{i}_r) = \ldots \quad \text{and}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{dr}{dt}\hat{i}_r + r\frac{d\theta}{dt}\hat{i}_\theta\right) = \ldots$$

using the derivative <u>product rule</u>.

⊛ Be sure not to skip steps!

② <u>"Inverse Newton's Laws"</u>

Ask yourself: "Am I given the motion?"

This means either $\{r(t), \theta(t)\}$ or
$\{r(t), \omega(t)\}$ or
$\{r(t), \alpha(t)\}$.

From any one of those combinations, you can compute <u>everything</u> that enters into the definitions of

$$\vec{v} = \frac{dr}{dt}\hat{i}_r + r\frac{d\theta}{dt}\hat{i}_\theta \quad \text{and}$$

$$\vec{a} = \left(\frac{d^2r}{dt^2} - r\omega^2\right)\hat{i}_r + \left(r\alpha + 2\omega\frac{dr}{dt}\right)\hat{i}_\theta.$$

You might need to use

$$\omega(t) = \int \alpha(t)\,dt \quad \text{and}$$

$$\theta(t) = \int \omega(t)\,dt.$$

Once $\vec{v}$ and $\vec{a}$ are known, you can compute many quantities:

$$\vec{F}_{total} = m\vec{a}$$

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times (m\vec{a})$$

$$= m(r\hat{\imath}_r) \times (a_r \hat{\imath}_r + a_\theta \hat{\imath}_\theta) = mra_\theta [\hat{\imath}_r \times \hat{\imath}_\theta]$$

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times (m\vec{v})$$

$$= m(r\hat{\imath}_r) \times (v_r \hat{\imath}_r + v_\theta \hat{\imath}_\theta) = mrv_\theta [\hat{\imath}_r \times \hat{\imath}_\theta]$$

$$= mr^2\omega [\hat{\imath}_r \times \hat{\imath}_\theta]$$

* In many cases, you will need to relate the total force you get from $\vec{F}_{total} = m\vec{a}$ to an underlying <u>fundamental</u> <u>force</u> (e.g., gravity, friction, etc.).

(i) Friction is responsible for cars making turns, for boxes staying on conveyor belts, etc.

(ii) Tension in a string might be responsible for the motion of a swinging object.

(iii) Gravity underlying cause of a satellite's motion, moon's orbit around Earth, etc.

Once you have identified physical force causing an object's motion, just equate it to

$$\vec{F}_{total} = m\vec{a} = m\left(\frac{d^2r}{dt^2} - r\omega^2\right)\hat{i}_r + m\left(r\alpha + 2\omega\frac{dr}{dt}\right)\hat{i}_\theta .$$

⊛ <u>Important point</u>: If you are given $r(t)$ but nothing about $\omega(t)$, you might need to use angular momentum conservation (see below) to determine $\omega(t)$.

3 Work and potential energy

Compute work done by a force as

$$W = \int \vec{F} \cdot d\vec{r} = \int \vec{F} \cdot \left(dr\, \hat{\imath}_r + r\, d\theta\, \hat{\imath}_r \right)$$

If $\vec{F}(r) = F(r)\, \hat{\imath}_r$, then $\vec{F} \cdot d\vec{r} = F(r)\, dr$

(since $\hat{\imath}_r \cdot \hat{\imath}_\theta = 0$)

and

$$\boxed{W = \int_{r_1}^{r_2} F(r)\, dr}$$

Likewise, the potential energy is

$$\boxed{U = -\int F(r)\, dr} \, .$$

4 Conservation of angular momentum

Two types of angular momentum:

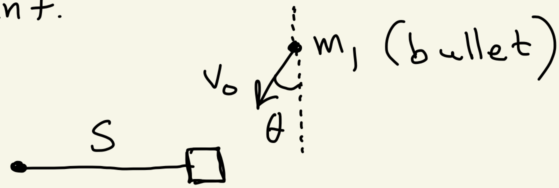
(i) point particle : $\vec{L} = \vec{r} \times \vec{p}$

object's position vector

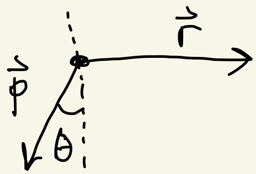
object's momentum vector

* Note that a freely-moving particle has constant $\vec{L}$, so calculate $\vec{r} \times \vec{p}$ anywhere it is most convenient.

Example:



$\vec{L}_1 = \vec{r} \times \vec{p}$ just before impact is



$\vec{L}_1 = s(m_1 v_0)\sin(90^\circ + \theta)$ $\otimes$ into page

(ii) Rigid body angular momentum:

$\vec{L} = I\omega$ (curl fingers in direction of ω, then thumb points in direction of $\vec{L}$)

Note that for a point mass, $\boxed{I = mr^2}$

If you have multiple objects, the total moment of inertia is just the sum

$$I = I_1 + I_2 + \cdots$$

Conservation of angular momentum is often used to find $\omega(t)$:

$$\boxed{I_0 \omega_0 = I(t)\,\omega(t)}$$

Changing moment of inertia causes changing angular velocity (figure skater).